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Analysis of Financial Investment Stock Markets Based on Linguistic Cubic Modified Fuzzy Decision-Making Strategies

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ABSTRACT

Pythagorean fuzzy set (PFS) and their modified frameworks deal with uncertainty more flexibly than the intuitionistic fuzzy set (IFS) frameworks. The existing frameworks address only partial aspects of real-life uncertainty, such as real-valued, interval-valued or complex-valued information. The existing models integrating interval-valued or complex-valued information lack of capable to represent linguistic uncertainty while the fuzzy set (FS) models based on linguistic term lack the capability to present complex-valued or higher order real-life uncertainty. Therefore, the existing techniques fail to simultaneously model interval imprecision, linguistic vagueness, and complex-valued cubic uncertainty within a unified framework. To overcome these limitations, the newly defined framework, called linguistic complex cubic Pythagorean fuzzy set (LCCuPFS) is demonstrated, integrating complex cubic Pythagorean fuzzy sets (CCuPFSs) with linguistic feature, providing a more realistic and robust representation of multifaceted uncertainty in daily real-life applications. Some basic operational laws and their corresponding aggregation operators (AOs) such as averaging AOs, geometric AOs, weighted averaging AOs, and weighted geometric AOs within the framework of LCCuPFSs are established. The key properties of the proposed AOs are investigated. Furthermore, a decision-making (DM) algorithm based on the newly defined approaches is developed. We employ the proposed techniques to establish a DM approach for solving real-world problems. To verify its practical utility, we apply the newly defined approaches in a real-world financial investment DM problem, where assessments are inherently imprecise, linguistically expressed, and influenced by interval-valued and complex-valued uncertainties. Finally, a comparative study between the proposed and existing approaches is investigated based on the ranking-wise performance and characteristic-wise evaluation.

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1. Introduction

The introduction is divided into seven major subsections. Theses subsections are financial investment, FSs and their extensions, cubic FSs (CuFSs) and their extensions, linguistic IFSs (LIFSs) and their extensions, research gaps and motivations, major contributions, and summary of the manuscript.

1.1 Financial Investment

Generating income or capital gain is the primary motive behind commitment of funds by investors into various types of assets that are either in the form of debt or equity instruments – a process commonly known as Financial Investment. This process is quite opposite to that of “Spending” which is primarily aimed at fulfilling an immediate need. Investment is future-oriented, thus specific returns can never ever be guaranteed thereby it involves a fair degree of risk. Commonly available options for financial investment consist of stocks, bonds, mutual funds, derivatives and crypto-currencies etc. Each alternative with respect to investment carries varying levels of risk, profitability, liquidity, and time horizon, thereby making investment a complex process which is and should be based on vigilant planning and evaluation.

Decision making process plays an important role in financial investment, due to the fact that investors are required to select the best option among various alternatives under states of uncertainty. A structured decision-making process helps in not only identifying investment opportunities, but also facilitates in defining evaluation criteria (i.e., risk, expected return, liquidity, and time horizon), and then trade-offs between them can be assessed accurately (Liu et al., 2024). For example, risk-seeking investors often prefer high-growth firms that may sometimes offer volatile returns, while on the other hand risk-averse investors usually prefer companies having a history of paying stable dividend. Risk management is significantly affected by decision-making, for instance when investors diversify their portfolios; they use hedging techniques and try to attain long-term financial goals such as retirement planning, wealth preservation or capital growth with minimal risk. Furthermore, systematic decision-making ensures rational investment choices supported by financial analyses through forecasting tools and at the same time it reduces emotional biases like fear and greed. FS (Zadeh, 1965) theory plays a critical role in financial investment. De Souza (de Souza, 2025) discussed risk evaluation in financial investment using fuzzy logic. Rahadian et al. (Rahadian et al., 2025) employed fuzzy DM technique to discuss stock market efficiency. Kaviyarasu et al. (Kaviyarasu et al., 2025) demonstrated exponential FSs and discussed their applications in investment DM based weighted mean procedure. Yuksel et al. (Yüksel et al., 2025) employed fuzzy DM approach to present novel financial solutions for sustainable investments. Idris et al. (Idris et al., 2025) discussed fuzzy AHP procedures for investment DM problems.

1.2 Fuzzy Sets and their Extensions

To better deal with uncertainty, vagueness and imprecision in daily real-life problems, Zadeh established the framework of FSs (Zadeh, 1965). The FS model generalized the framework of crisp set, described by a membership (MS) function that assigns each element of universal set a numerical value in the closed interval [0,1]. FSs play a critical role in many DM problems such as MCDM, classification, pattern recognition, artificial intelligence (AI) and medical diagnosis problems (Kumar & Garg, 2025; Miliauskaitė & Kalibatiene, 2025; Saqlain, 2025; Singh & Bhardwaj, 2025). The IFS (Atanassov, 1999) framework extends the notion of FSs by introducing a non-membership (NMS) degree. IFSs have been successfully applied in many real-world uncertainty problems. Khan et al. (Khan et al., 2025) demonstrated some innovative Dombi AOs for IFSs and employed them for handling real-life problems. Fahmi et al. (Fahmi et al., 2018) introduced aggregation procedure for renewable source energy selection within the framework of IFSs. Unver developed a DM approach based on intuitionistic fuzzy Gaussian AOs (Ünver, 2025). The PFS (Yager, 2013) framework, proposed by Yager, relaxes the restriction of IFSs, allowing greater flexibility in presenting uncertainty. They are very useful in MCDM problems. Asif et al. (Asif et al., 2025) demonstrated Hamacher AOs for DM process within the framework of PFSs. Palanikumar et al. (Palanikumar, Kausar, Pamucar, et al., 2025) discussed industrial robot selection problems based on PF normal AOs. Interval-valued PFSs (IVPFSs) (Liang et al., 2018) improve the PFS model by enabling the MS and NMS degrees to be described as intervals rather than single values. They are a more successful model in real-life problems. Biswas and Dey solved DM problems using newly defined AOs within the IVIFS framework (Biswas & Dey, 2025). Kumar et al. (Kumar et al., 2025) discussed real-life problems using IVPFSs with DM approaches. Hu et al. (Hu et al., 2025) established an innovative valuation procedure using IVPFSs. A Complex IVIFSs (CIVIFSs) (Chinnadurai, Thayalan, & Bobin, 2021) simultaneously provides oscillatory or periodic information (via complex phases) and amplitude uncertainty (via intervals) while preserving the Pythagorean flexibility**.** This makes it suitable for modeling time-dependent and complex decision environments. Yazbek et al. (Yazbek et al., 2023) discussed economics characteristics of an institution employing CIVPFS framework. Ali et al. (Ali et al., 2021) employed CIVPFSs to propose Einstein AOs for solving DM problems. Palanikumar et al. (Palanikumar, Kausar, Tharaniya, et al., 2025) solved real-world problems using some generalized notion of CIVPFSs.

1.3 Cubic Fuzzy Sets and their Extensions

An advanced extension of FS theory is the CuFSs (Jun et al., 2011), combining the notions of FSs and interval-valued FSs into a single framework. The CuFS model is used to handle two layers of uncertainty: precise and interval-based uncertainty. CuFSs are successfully applicable in many DM problems. Shi et al. (Shi et al., 2022) developed some innovative cubic fuzzy graphs for solving real-life applications. Fahmi et al. (Fahmi et al., 2018) employed CuFSs to proposed Einstein AOs. They applied them successfully in real-world DM problems. An advanced extension of CuFSs, known as Cubic IFSs (CuIFSs) (Kaur & Garg, 2018), was established by Kaur and Garg, integrating the notions of IFSs and IVIFSs for presenting richer modeling of uncertainty in DM problems. Garg and Kaur established an innovative DM procedure using correlation coefficients within the CuIFS environment (Garg & Kaur, 2022). Priyadharshini et al. (Priyadharshini et al., 2025) employed CuIFSs to demonstrate some novel similarity measures and applied them for solving real-life problems. Chunsong et al. (Chunsong et al., 2024) presented Schweizer and Sklar power AOs using CuIFSs. They solved a DM problem based on the newly defined approaches. Cubic PFSs (CuPFSs) (Abbas et al., 2019) integrates the frameworks of PFSs and IVPFSs, providing greater flexibility for modeling complex DM problems.

Amin et al. (Amin et al., 2022) established an innovative aggregation procedure for solving DM problems using CuPFSs. Rahim et al. (Rahim et al., 2022) proposed Bonferroni operators within the CuPFS framework. They employed them for solving selection problems. Rahim et al. (Rahim et al., 2023) employed CuPFSs to establish Dombi AOs for solving real-world DM problems. Later on, Chinnadurai et al. introduced complex CuPFSs (CCuPFSs) (Chinnadurai, Thayalan, & Rao, 2021), integrating interval-valued multi-dimensional and complex information. The CCuPFS framework provides a comprehensive mathematical structure for modeling highly uncertain and dynamic decision environments**.**

1.4 Linguistic Intuitionistic Fuzzy Sets and their Extensions

The LIFS (Zhang, 2014) framework is an advanced generalization of IFSs established to deal with DM problems in which the information is described in qualitative and quantitative rather than precise numerical values. In LIFSs, the MS and NMS degrees are expressed based on linguistic terms. They are more suited model for handling real-world DM problems. Alghazzawi et al. (Alghazzawi et al., 2025) demonstrated Dombi AOs within the LIFS framework. They proposed an innovative DM procedure using the newly defined approaches. Ameer et al. (Ameer et al., 2025) established a new approach of DM for selection problems using LIFSs. Peter Dawson and Selvaraj presented VIKOR technique for artificial neural network under LIFSs (Peter Dawson & Selvaraj, 2025). Malik et al. (Malik et al., 2023) presented some new approaches for DM problems using LIFSs. Robinson and Leonishiya developed TOPSIS approach within the LIFS environment (Robinson & Leonishiya, 2025).

The linguistic PFS (LPFS) (Garg, 2018) model generalized the framework of LIFSs by providing by requiring the transformed numerical values to fulfill the Pythagorean condition. They have been successfully employed in many DM problems. Kalsoom et al. (Kalsoom et al., 2026) developed a new DM approach using Dombi AOs within the LPFS framework. Ping et al. (Ping et al., 2022) employed LPFSs to establish an innovative technique for real-world applications. Wang et al. (Wang et al., 2021) solved DM problems in the environment of LPFSs. Garg and Kumar (Garg & Kumar, 2019) proposed a new model called linguistic IVIFSs (LIVIFSs), integrating interval numbers in linguistic MS and NMS functions. The LIVIFS model is successfully applied in many DM problems. Qin et al. (Qin et al., 2020) introduced new AOs using LIVIFSs and employed them for solving DM problems. Xu et al. (Xu et al., 2021) discussed aggregation procedure for real-world problems based on LIVIFSs. Later on, Garg extended the notion of LIVIFSs to LIVPFSs by providing Pythagorean flexibility in describing uncertainty based on linguistic information (Garg, 2020). He established an innovative DM procedure based on the newly defined approaches. Khan et al. (Khan et al., 2024) demonstrated the linguistic Hesitant fuzzy rough set models and applications. They defined AOs within the framework of LCIFSs and used them for addressing DM problems. Recent contributions on fuzzy decision can be read in (Ali et al., 2024) and (Liu et al., 2025).

1.5 Research Gaps and Motivations

The PFS framework offers greater flexibility than the IFS model by enabling a wider domain for MS and NMS functions; however, this framework is limited to real-world applications and cannot deal with periodic or complex uncertainty. The complex PFS (CPFS) model was proposed, allowing the representation of periodic or complex uncertainty based on amplitude and phase terms. But this model cannot deal with interval-valued information or linguistic vagueness. The IVPFS framework generalize the PFS model by integrating interval-valued information to better handle uncertainty information; however, this framework fails to capture complex-valued information and linguistic evaluations in DM problems. The CIVPFS model integrates both complex-valued and interval-valued information; however, they cannot contribute qualitative linguistic expressions. The LIFS and LPFSs models are comprehensive for dealing with qualitative and uncertain information. But they cannot effectively present complex-valued and interval-valued information. LIVPFSs further improve LIFSs and LPFSs by describing both MS and NMS degrees as interval-valued linguistic term with Pythagorean flexibility. But the LIVPFS framework cannot effectively handle complex-valued information. LCIFSs enhance the LIFS model by enabling the MS and NMS degrees to be described as complex-valued linguistic information. But the LCIFS framework cannot effectively handle interval-valued information. All of these frameworks are limited to real-life applications. Therefore, there exists a clear research gap in introducing a unified model that simultaneously integrates linguistic terms, interval-valued uncertainty, complex-valued information, and cubic Pythagorean structures. Motivated by this gap, we propose a new approach involving four characteristics including linguistic feature, interval-valued MS and NMS degrees, complex-valued information and Pythagorean flexibility.

1.6 Major Contributions

AOs play a pivotal role in the newly defined framework that integrates CCuPF information with linguistic assessment. The proposed approach involves linguistic feature, interval-valued MS and NMS degrees, complex-valued information and Pythagorean flexibility, the AOs based on the newly defined model contribute a major role in real-world DM problems. The main contributions of the proposed study are listed below:

1) To introduce a new model, called LCCuPFSs, integrating the frameworks of CCuPFSs and linguistic information.

2) To establish set theoretic operations for newly defined model.

3) To demonstrate averaging AOs, geometric AOs, weighted averaging AOs, and weighted geometric AOs within the LCCuPFS framework.

4) To investigate their key properties.

5) To develop a DM approach using the proposed model and AOs.

6) To investigate the practical utility of the DM approach in real-world applications.

7) To verify its effectiveness, demonstrate the comparison between the newly defined techniques and existing techniques.

1.7 Summary of the Manuscript

This study is divided as in the following sections.

Section 2 offers the fundamental frameworks of FS theory.

Section 3 introduces the proposed model and key set theoretic operations.

Section 4 provides basic operation laws for LCCuPFSs.

Section 5 introduces averaging AOs, geometric AOs, weighted averaging AOs, and weighted geometric AOs within the LCCuPFS framework. Further, their basic properties are investigated.

Section 6 develop a DM algorithm based the newly defined approaches.

Section 7 discusses their practical utility in real-life applications.

Section 8 provides a comparative study between the newly defined techniques and existing techniques.

Section 9 offers the conclusion of the proposed study.

2. Preliminaries

This section provides some fundamental concepts of FS theory.

Definition 1: (Herrera & Martínez, 2001) A set $\mathbb{S} = \{\mathfrak{s}_t | t = 0,1,2,\dots,\hbar\}$ with odd cardinality is said to be linguistic term set (LTS) where $\mathfrak{s}_t$ having the following characteristics.

1). $\mathfrak{s}_k \leq \mathfrak{s}_t \Leftrightarrow k \leq t$.

2). Negation$(\mathfrak{s}_k) = \mathfrak{s}_{\hbar - k}$.

3). $\max\{\mathfrak{s}_k, \mathfrak{s}_t\} = \mathfrak{s}_{\max\{k,t\}}$.

4). $\min\{\mathfrak{s}_k, \mathfrak{s}_t\} = \mathfrak{s}_{\min\{k,t\}}$.

Xu (Xu, 2004) generalized this framework to continuous LTS and demonstrated as:

$$\mathbb{S}_{[0,\hbar]} = \{\mathfrak{s}_t | \mathfrak{s}_0 \leq \mathfrak{s}_t \leq \mathfrak{s}_\hbar;\ t \in [0,\hbar]\}$$

Note: The symbol "Ł" indicates the set in general, but its structure differs depending on the fuzzy models.

Definition 2: (Zhang, 2014) An LIFS "Ł" in U is demonstrated as:

$$Ŀ = \{(ǎ, (s_{A_{Ŀ}(ǎ)}, s_{B_{Ŀ}(ǎ)}): ǎ \in U)\},$$

where $s_{A_{Ŀ}(ǎ)}, s_{B_{Ŀ}(ǎ)} \in \mathbb{S}_{[0,\hbar]}$ denote the linguistic MS and NMS degrees of ǎ respectively such that they satisfy the condition:

$$A_{Ŀ}(ǎ) + B_{Ŀ}(ǎ) \leq \hbar, \qquad \forall\, ǎ \in U.$$

Definition 3: (Garg, 2018) An LPFS "Ŀ" in U is demonstrated as:

$$Ŀ = \{(ǎ, (s_{A_{Ŀ}(ǎ)}, s_{B_{Ŀ}(ǎ)}): ǎ \in U)\},$$

where $s_{A_{Ŀ}(ǎ)}, s_{B_{Ŀ}(ǎ)} \in \mathbb{S}_{[0,\hbar]}$ denote the linguistic MS and NMS degrees of ǎ respectively such that they satisfy the condition:

$$(A_{Ŀ}(ǎ))^2 + (B_{Ŀ}(ǎ))^2 \leq \hbar^2, \forall\, ǎ \in U.$$

Definition 4: (Garg & Kumar, 2019) An LIVIFS "Ŀ" in U is demonstrated as:

$$Ŀ = \{(ǎ, (s_{A_{Ŀ}(ǎ)}, s_{B_{Ŀ}(ǎ)}): ǎ \in U)\},$$

where $s_{A_{Ŀ}(ǎ)} = [s_{a_{Ŀ}(ǎ)}, s_{b_{Ŀ}(ǎ)}] \in [s_0, s_\hbar], s_{B_{Ŀ}(ǎ)} = [s_{c_{Ŀ}(ǎ)}, s_{d_{Ŀ}(ǎ)}] \in [s_0, s_\hbar]$ denote the linguistic MS and NMS degrees of ǎ respectively such that they satisfy the condition:

$$b_{Ŀ}(ǎ) + d_{Ŀ}(ǎ) \leq \hbar, \forall\, ǎ \in U.$$

Definition 5: (Garg, 2020) An LIVPFS "Ŀ" in U is demonstrated as:

$$Ŀ = \{(ǎ, (s_{A_{Ŀ}(ǎ)}, s_{B_{Ŀ}(ǎ)}): ǎ \in U)\},$$

where $s_{A_{Ŀ}(ǎ)} = [s_{a_{Ŀ}(ǎ)}, s_{b_{Ŀ}(ǎ)}] \in [s_0, s_\hbar], s_{B_{Ŀ}(ǎ)} = [s_{c_{Ŀ}(ǎ)}, s_{d_{Ŀ}(ǎ)}] \in [s_0, s_\hbar]$ denote the linguistic MS and NMS degrees of ǎ respectively such that they satisfy the condition:

$$(b_{Ŀ}(ǎ))^2 + (d_{Ŀ}(ǎ))^2 \leq \hbar^2, \forall\, ǎ \in U.$$

Definition 6: (Khan et al., 2024) An LCIFS "Ŀ" in U is demonstrated as:

$$Ŀ = \{(ǎ, (s_{A_{Ŀ}(ǎ)}, s_{B_{Ŀ}(ǎ)}): ǎ \in U)\},$$

where $s_{A_{Ŀ}(ǎ)} = s_{u_{Ŀ}(ǎ)} e^{i s_{\mathfrak{H}_{Ŀ}(ǎ)}}, s_{B_{Ŀ}(ǎ)} = s_{v_{Ŀ}(ǎ)} e^{i s_{n_{Ŀ}(ǎ)}}$ denote the linguistic MS and NMS degrees of ǎ respectively such that they satisfy the conditions:

$$(u_{Ŀ}(ǎ))^2 + (v_{Ŀ}(ǎ))^2 \leq \hbar^2 \ \& \ (\mathfrak{H}_{Ŀ}(ǎ))^2 + (n_{Ŀ}(ǎ))^2 \leq \hbar^2, \forall\, ǎ \in U.$$

Definition 7: (Jun et al., 2011) A CuFS Ŀ in U is demonstrated as:

$$Ŀ = \{ǎ, (A_{Ŀ}(ǎ), B_{Ŀ}(ǎ)): ǎ \in U\}$$

$$Ŀ = \{ǎ, ([a_{Ŀ}(ǎ), b_{Ŀ}(ǎ)], B_{Ŀ}(ǎ)): ǎ \in U\},$$

where $A_{Ŀ}(ǎ) = [a_{Ŀ}(ǎ), b_{Ŀ}(ǎ)]$ and $B_{Ŀ}(ǎ)$ denote the framework of IVFSs and FSs, respectively.

Definition 8: (Kaur & Garg, 2018) A CuIFS Ŀ in U is demonstrated as:

$$Ŀ = \{ǎ, (A_{Ŀ}(ǎ), B_{Ŀ}(ǎ)): ǎ \in U\}$$

$$Ŀ = \{ǎ, (([a_{Ŀ}(ǎ), b_{Ŀ}(ǎ)], [c_{Ŀ}(ǎ), d_{Ŀ}(ǎ)]), (u_{Ŀ}(ǎ), v_{Ŀ}(ǎ))): ǎ \in U\},$$

where $A_{Ŀ}(ǎ) = ([a_{Ŀ}(ǎ), b_{Ŀ}(ǎ)], [c_{Ŀ}(ǎ), d_{Ŀ}(ǎ)])$ and $B_{Ŀ}(ǎ) = (u_{Ŀ}(ǎ), v_{Ŀ}(ǎ))$ denote the framework of IVIFSs and IFSs, respectively such that they satisfy the conditions:

$$0 \leq b_{Ŀ}(ǎ) + d_{Ŀ}(ǎ) \leq 1, \ \& \ 0 \leq u_{Ŀ}(ǎ) + v_{Ŀ}(ǎ) \leq 1, \forall\, ǎ \in U.$$

Definition 9: (Jun et al., 2018) A CuIVIFS Ŀ in U is demonstrated as:

$$Ŀ = \{ǎ, (A_{Ŀ}(ǎ), B_{Ŀ}(ǎ)): ǎ \in U\}$$

$$\mathcal{L} = \left\{\check{a}, \left(\left([\mathbb{a}_{\mathcal{L}}(\check{a}), \mathbb{b}_{\mathcal{L}}(\check{a})], [\mathbb{c}_{\mathcal{L}}(\check{a}), \mathbb{d}_{\mathcal{L}}(\check{a})]\right), \mathbb{B}_{\mathcal{L}}(\check{a})\right): \check{a} \in U\right\},$$

where $\mathbb{A}_{\mathcal{L}}(\check{a}) = ([\mathbb{a}_{\mathcal{L}}(\check{a}), \mathbb{b}_{\mathcal{L}}(\check{a})], [\mathbb{c}_{\mathcal{L}}(\check{a}), \mathbb{d}_{\mathcal{L}}(\check{a})])$ and $\mathbb{B}_{\mathcal{L}}(\check{a})$ represent the framework of IVIFSs and FSs, respectively.

Definition 10: (Abbas et al., 2019) A CuPFS $\mathcal{L}$ in U is demonstrated as:

$$\mathcal{L} = \{\check{a}, (\mathbb{A}_{\mathcal{L}}(\check{a}), \mathbb{B}_{\mathcal{L}}(\check{a})): \check{a} \in U\}$$

$$\mathcal{L} = \left\{\check{a}, \left(\left([\mathbb{a}_{\mathcal{L}}(\check{a}), \mathbb{b}_{\mathcal{L}}(\check{a})], [\mathbb{c}_{\mathcal{L}}(\check{a}), \mathbb{d}_{\mathcal{L}}(\check{a})]\right), (\mathfrak{u}_{\mathcal{L}}(\check{a}), \mathfrak{v}_{\mathcal{L}}(\check{a}))\right): \check{a} \in U\right\},$$

where $\mathbb{A}_{\mathcal{L}}(\check{a}) = ([\mathbb{a}_{\mathcal{L}}(\check{a}), \mathbb{b}_{\mathcal{L}}(\check{a})], [\mathbb{c}_{\mathcal{L}}(\check{a}), \mathbb{d}_{\mathcal{L}}(\check{a})])$ and $\mathbb{B}_{\mathcal{L}}(\check{a}) = (\mathfrak{u}_{\mathcal{L}}(\check{a}), \mathfrak{v}_{\mathcal{L}}(\check{a}))$ denote the framework of IVPFSs and PFSs, respectively, such that they satisfy the conditions:

$$0 \le \left(\mathbb{b}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathbb{d}_{\mathcal{L}}(\check{a})\right)^2 \le 1, \ \& \ 0 \le \left(\mathfrak{u}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathfrak{v}_{\mathcal{L}}(\check{a})\right)^2 \le 1, \forall\, \check{a} \in U.$$

Definition 11: (Chinnadurai, Thayalan, & Rao, 2021) A CCuPFS $\mathcal{L}$ in U is demonstrated as:

$$\mathcal{L} = \{\check{a}, (\mathbb{A}_{\mathcal{L}}(\check{a}), \mathbb{B}_{\mathcal{L}}(\check{a})): \check{a} \in U\}$$

$$\mathcal{L} = \left\{\check{a}, \begin{pmatrix} \langle [\mathbb{a}_{\mathcal{L}}(\check{a}), \mathbb{b}_{\mathcal{L}}(\check{a})] e^{i2\pi[\alpha_{\mathcal{L}}(\check{a}), \beta_{\mathcal{L}}(\check{a})]}, [\mathbb{c}_{\mathcal{L}}(\check{a}), \mathbb{d}_{\mathcal{L}}(\check{a})] e^{i2\pi[\gamma_{\mathcal{L}}(\check{a}), \delta_{\mathcal{L}}(\check{a})]} \rangle, \\ \langle \mathfrak{u}_{\mathcal{L}}(\check{a}) e^{i2\pi \text{Ѣ}_{\mathcal{L}}(\check{a})}, \mathfrak{v}_{\mathcal{L}}(\check{a}) e^{i2\pi \mathfrak{n}_{\mathcal{L}}(\check{a})} \rangle \end{pmatrix}: \check{a} \in U\right\},$$

where $\mathbb{A}_{\mathcal{L}}(\check{a})$ and $\mathbb{B}_{\mathcal{L}}(\check{a})$ represent the framework of CIVPFSs and CPFSs, respectively, such that they satisfy the conditions:

$$0 \le \left(\mathbb{b}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathbb{d}_{\mathcal{L}}(\check{a})\right)^2 \le 1,\ 0 \le \left(\beta_{\mathcal{L}}(\check{a})\right)^2 + \left(\delta_{\mathcal{L}}(\check{a})\right)^2 \le 1,\ 0 \le \left(\mathfrak{u}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathfrak{v}_{\mathcal{L}}(\check{a})\right)^2 \le 1, \text{ and } 0 \le \left(\text{Ѣ}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathfrak{n}_{\mathcal{L}}(\check{a})\right)^2 \le 1, \forall\, \check{a} \in U..$$

3. Linguistic Complex Cubic Pythagorean Fuzzy Sets

This section aims to formulate a new mathematical framework, called LCCuPFSs, which assigns CCuPFSs to linguistic terms. This newly mathematical framework involves four characteristics including linguistic feature, interval-valued MS and NMS degrees, complex-valued information and Pythagorean flexibility. Furthermore, various operations of set theory are demonstrated for the newly defined model.

Definition 12: Let U denotes a UoD. An LCCuPFS $\mathcal{L}$ on $\mathbb{S}_{[0,\hbar]}$ is introduced as:

$$\mathcal{L} = \left\{\check{a}, \left(\mathfrak{s}_{\mathbb{A}_{\mathcal{L}}(\check{a})}, \mathfrak{s}_{\mathbb{B}_{\mathcal{L}}(\check{a})}\right): \check{a} \in U\right\},$$

where $\mathfrak{s}_{\mathbb{A}_{\mathcal{L}}(\check{a})}$ and $\mathfrak{s}_{\mathbb{B}_{\mathcal{L}}(\check{a})}$ represent the following mathematical expressions, respectively.

$$\mathbb{A}_{\mathcal{L}}(\check{a}) = \left(\left[\mathfrak{s}_{\mathbb{a}_{\mathcal{L}}(\check{a})}, \mathfrak{s}_{\mathbb{b}_{\mathcal{L}}(\check{a})}\right] e^{i\left[s_{\alpha_{\mathcal{L}}(\check{a})}, s_{\beta_{\mathcal{L}}(\check{a})}\right]}, \left[\mathfrak{s}_{\mathbb{c}_{\mathcal{L}}(\check{a})}, \mathfrak{s}_{\mathbb{d}_{\mathcal{L}}(\check{a})}\right] e^{i\left[s_{\gamma_{\mathcal{L}}(\check{a})}, s_{\delta_{\mathcal{L}}(\check{a})}\right]}\right),$$

and

$$\mathbb{B}_{\mathcal{L}}(\check{a}) = \left(\mathfrak{s}_{\mathfrak{u}_{\mathcal{L}}(\check{a})} e^{i s_{\text{Ѣ}_{\mathcal{L}}(\check{a})}}, \mathfrak{s}_{\mathfrak{v}_{\mathcal{L}}(\check{a})} e^{i s_{\mathfrak{n}_{\mathcal{L}}(\check{a})}}\right).$$

Thus, the LCCuPFS model can be expressed as:

$$\mathcal{L} = \left\{\left(\check{a}, \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_{\mathcal{L}}(\check{a})}, \mathfrak{s}_{\mathbb{b}_{\mathcal{L}}(\check{a})}\right] e^{i\left[s_{\alpha_{\mathcal{L}}(\check{a})}, s_{\beta_{\mathcal{L}}(\check{a})}\right]}, \left[\mathfrak{s}_{\mathbb{c}_{\mathcal{L}}(\check{a})}, \mathfrak{s}_{\mathbb{d}_{\mathcal{L}}(\check{a})}\right] e^{i\left[s_{\gamma_{\mathcal{L}}(\check{a})}, s_{\delta_{\mathcal{L}}(\check{a})}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_{\mathcal{L}}(\check{a})} e^{i s_{\text{Ѣ}_{\mathcal{L}}(\check{a})}}, \mathfrak{s}_{\mathfrak{v}_{\mathcal{L}}(\check{a})} e^{i s_{\mathfrak{n}_{\mathcal{L}}(\check{a})}}\right) \end{pmatrix}\right): \check{a} \in U\right\},$$

where:

$$0 \le \left(\mathbb{b}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathbb{d}_{\mathcal{L}}(\check{a})\right)^2 \le \hbar^2,\ 0 \le \left(\beta_{\mathcal{L}}(\check{a})\right)^2 + \left(\delta_{\mathcal{L}}(\check{a})\right)^2 \le \hbar^2,\ 0 \le \left(\mathfrak{u}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathfrak{v}_{\mathcal{L}}(\check{a})\right)^2 \le \hbar^2, \text{ and } 0 \le \left(\text{Ѣ}_{\mathcal{L}}(\check{a})\right)^2 + \left(\mathfrak{n}_{\mathcal{L}}(\check{a})\right)^2 \le \hbar^2.$$

Definition 13: A pair $\mathcal{L} = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_{\mathcal{L}}(\check{a})}, \mathfrak{s}_{\mathbb{b}_{\mathcal{L}}(\check{a})}\right] e^{i\left[s_{\alpha_{\mathcal{L}}(\check{a})}, s_{\beta_{\mathcal{L}}(\check{a})}\right]}, \left[\mathfrak{s}_{\mathbb{c}_{\mathcal{L}}(\check{a})}, \mathfrak{s}_{\mathbb{d}_{\mathcal{L}}(\check{a})}\right] e^{i\left[s_{\gamma_{\mathcal{L}}(\check{a})}, s_{\delta_{\mathcal{L}}(\check{a})}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_{\mathcal{L}}(\check{a})} e^{i s_{\text{Ѣ}_{\mathcal{L}}(\check{a})}}, \mathfrak{s}_{\mathfrak{v}_{\mathcal{L}}(\check{a})} e^{i s_{\mathfrak{n}_{\mathcal{L}}(\check{a})}}\right) \end{pmatrix}$ is said to be linguistic complex cubic Pythagorean fuzzy number (LCCuPFN) where $\mathbb{a}_{Ƚ}(ă), \mathbb{b}_{Ƚ}(ă), \mathbb{c}_{Ƚ}(ă), \mathbb{d}_{Ƚ}(ă), \alpha_{Ƚ}(ă), \beta_{Ƚ}(ă), \gamma_{Ƚ}(ă), \delta_{Ƚ}(ă), \mathfrak{u}_{Ƚ}(ă), \mathfrak{v}_{Ƚ}(ă), Ֆ_{Ƚ}(ă), \mathfrak{n}_{Ƚ}(ă) \in [0, \hbar]$, and $0 \leq (\mathbb{b}_{Ƚ}(ă))^2 + (\mathbb{d}_{Ƚ}(ă))^2 \leq \hbar^2$, $0 \leq (\beta_{Ƚ}(ă))^2 + (\delta_{Ƚ}(ă))^2 \leq \hbar^2$, $0 \leq (\mathfrak{u}_{Ƚ}(ă))^2 + (\mathfrak{v}_{Ƚ}(ă))^2 \leq \hbar^2$, and $0 \leq (Ֆ_{Ƚ}(ă))^2 + (\mathfrak{n}_{Ƚ}(ă))^2 \leq \hbar^2$.

Note: For simplification, we may use $Ƚ_i = \begin{pmatrix} \left([s_{\mathbb{a}_i}, s_{\mathbb{b}_i}]e^{i[s_{\alpha_i}, s_{\beta_i}]}, [s_{\mathbb{c}_i}, s_{\mathbb{d}_i}]e^{i[s_{\gamma_i}, s_{\delta_i}]}\right), \\ \left(s_{\mathfrak{u}_i}e^{is_{Ֆ_i}}, s_{\mathfrak{v}_i}e^{is_{\mathfrak{n}_i}}\right) \end{pmatrix}$ for LCCuPFN.

Definition 14: Let $Ƚ_1 = \begin{pmatrix} \left([s_{\mathbb{a}_1}, s_{\mathbb{b}_1}]e^{i[s_{\alpha_1}, s_{\beta_1}]}, [s_{\mathbb{c}_1}, s_{\mathbb{d}_1}]e^{i[s_{\gamma_1}, s_{\delta_1}]}\right), \\ \left(s_{\mathfrak{u}_1}e^{is_{Ֆ_1}}, s_{\mathfrak{v}_1}e^{is_{\mathfrak{n}_1}}\right) \end{pmatrix}$ and $Ƚ_2 = \begin{pmatrix} \left([s_{\mathbb{a}_2}, s_{\mathbb{b}_2}]e^{i[s_{\alpha_2}, s_{\beta_2}]}, [s_{\mathbb{c}_2}, s_{\mathbb{d}_2}]e^{i[s_{\gamma_2}, s_{\delta_2}]}\right), \\ \left(s_{\mathfrak{u}_2}e^{is_{Ֆ_2}}, s_{\mathfrak{v}_2}e^{is_{\mathfrak{n}_2}}\right) \end{pmatrix}$ be two LCCuPFNs, then

1). $Ƚ_1 = Ƚ_2$ iff $\mathbb{a}_1 = \mathbb{a}_2, \mathbb{b}_1 = \mathbb{b}_2, \mathbb{c}_1 = \mathbb{c}_2, \mathbb{d}_1 = \mathbb{d}_2, \alpha_1 = \alpha_2, \beta_1 = \beta_2, \gamma_1 = \gamma_2, \delta_1 = \delta_2, \mathfrak{u}_1 = \mathfrak{u}_2, \mathfrak{v}_1 = \mathfrak{v}_2, Ֆ_1 = Ֆ_2$, and $\mathfrak{n}_1 = \mathfrak{n}_2$.

2). $Ƚ_1 \leq Ƚ_2$ iff $\mathbb{a}_1 \leq \mathbb{a}_2, \mathbb{b}_1 \leq \mathbb{b}_2, \mathbb{c}_1 \geq \mathbb{c}_2, \mathbb{d}_1 \geq \mathbb{d}_2, \alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \geq \gamma_2, \delta_1 \geq \delta_2, \mathfrak{u}_1 \leq \mathfrak{u}_2, \mathfrak{v}_1 \geq \mathfrak{v}_2, Ֆ_1 \leq Ֆ_2$, and $\mathfrak{n}_1 \geq \mathfrak{n}_2$.

3). ${Ƚ_1}^{\mathbb{c}} = \begin{pmatrix} \left([s_{\mathbb{c}_1}, s_{\mathbb{d}_1}]e^{i[s_{\gamma_1}, s_{\delta_1}]}, [s_{\mathbb{a}_1}, s_{\mathbb{b}_1}]e^{i[s_{\alpha_1}, s_{\beta_1}]}\right), \\ \left(s_{\mathfrak{v}_1}e^{is_{\mathfrak{n}_1}}, s_{\mathfrak{u}_1}e^{is_{Ֆ_1}}\right) \end{pmatrix}$ represent the complement $Ƚ_1$.

4). Let $Ƚ_1$ and $Ƚ_2$ be two LCCuPFSs over $\mathbb{S}_{[0,\hbar]}$. Then, the union operation within the framework of LCCuPFSs is defined as:

$$Ƚ_1 \cup Ƚ_2 = \begin{pmatrix} \left([\max(s_{\mathbb{a}_1}, s_{\mathbb{a}_2}), \max(s_{\mathbb{b}_1}, s_{\mathbb{b}_2})]e^{i[\max(s_{\alpha_1}, s_{\alpha_2}), \max(s_{\beta_1}, s_{\beta_2})]}, [\min(s_{\mathbb{c}_1}, s_{\mathbb{c}_2}), \min(s_{\mathbb{d}_1}, s_{\mathbb{d}_2})]e^{i[\min(s_{\gamma_1}, s_{\gamma_2}), \min(s_{\delta_1}, s_{\delta_2})]}\right), \\ \left(\max(s_{\mathfrak{u}_1}, s_{\mathfrak{u}_1})\, e^{imax(s_{Ֆ_1}, s_{Ֆ_2})}, \min(s_{\mathfrak{v}_1}, s_{\mathfrak{v}_2})\, e^{imin(s_{\mathfrak{n}_1}, s_{\mathfrak{n}_2})}\right) \end{pmatrix}$$
$$= \begin{pmatrix} \left([s_{\max(\mathbb{a}_1, \mathbb{a}_2)}, s_{\max(\mathbb{b}_1, \mathbb{b}_2)}]e^{i[s_{\max(\alpha_1, \alpha_2)}, s_{\max(\beta_1, \beta_2)}]}, [s_{\min(\mathbb{c}_1, \mathbb{c}_2)}, s_{\min(\mathbb{d}_1, \mathbb{d}_2)}]e^{i[s_{\min(\gamma_1, \gamma_2)}, s_{\min(\delta_1, \delta_2)}]}\right), \\ \left(s_{\max(\mathfrak{u}_1, \mathfrak{u}_2)}e^{is_{\max(Ֆ_1, Ֆ_2)}}, s_{\min(\mathfrak{v}_1, \mathfrak{v}_2)}e^{is_{\min(\mathfrak{n}_1, \mathfrak{n}_2)}}\right) \end{pmatrix}.$$

5). Let $Ƚ_1$ and $Ƚ_2$ be two LCCuPFSs over $\mathbb{S}_{[0,\hbar]}$. Then, the intersection operation within the framework of LCCuPFSs is defined as:

$$Ƚ_1 \cap Ƚ_2 = \begin{pmatrix} \left([\min(s_{\mathbb{a}_1}, s_{\mathbb{a}_2}), \min(s_{\mathbb{b}_1}, s_{\mathbb{b}_2})]e^{i[\min(s_{\alpha_1}, s_{\alpha_2}), \min(s_{\beta_1}, s_{\beta_2})]}, [\max(s_{\mathbb{c}_1}, s_{\mathbb{c}_2}), \max(s_{\mathbb{d}_1}, s_{\mathbb{d}_2})]e^{i[\max(s_{\gamma_1}, s_{\gamma_2}), \max(s_{\delta_1}, s_{\delta_2})]}\right), \\ \left(\min(s_{\mathfrak{u}_1}, s_{\mathfrak{u}_1})\, e^{imin(s_{Ֆ_1}, s_{Ֆ_2})}, \max(s_{\mathfrak{v}_1}, s_{\mathfrak{v}_2})\, e^{imax(s_{\mathfrak{n}_1}, s_{\mathfrak{n}_2})}\right) \end{pmatrix}$$
$$= \begin{pmatrix} \left([s_{\min(\mathbb{a}_1, \mathbb{a}_2)}, s_{\min(\mathbb{b}_1, \mathbb{b}_2)}]e^{i[s_{\min(\alpha_1, \alpha_2)}, s_{\min(\beta_1, \beta_2)}]}, [s_{\max(\mathbb{c}_1, \mathbb{c}_2)}, s_{\max(\mathbb{d}_1, \mathbb{d}_2)}]e^{i[s_{\max(\gamma_1, \gamma_2)}, s_{\max(\delta_1, \delta_2)}]}\right), \\ \left(s_{\min(\mathfrak{u}_1, \mathfrak{u}_2)}e^{is_{\min(Ֆ_1, Ֆ_2)}}, s_{\max(\mathfrak{v}_1, \mathfrak{v}_2)}e^{is_{\max(\mathfrak{n}_1, \mathfrak{n}_2)}}\right) \end{pmatrix}.$$

Theorem 1: Let $Ƚ_1, Ƚ_2$, and $Ƚ_3$ be three LCCuPFSs over $\mathbb{S}_{[0,\hbar]}$. Then,

i) $Ƚ_1 \cup Ƚ_2 = Ƚ_2 \cup Ƚ_1$,
ii) $Ƚ_1 \cap Ƚ_2 = Ƚ_2 \cap Ƚ_1$,
iii) $(Ƚ_1 \cup Ƚ_2) \cup Ƚ_3 = Ƚ_1 \cup (Ƚ_2 \cup Ƚ_3)$,
iv) $(Ƚ_1 \cap Ƚ_2) \cap Ƚ_3 = Ƚ_1 \cap (Ƚ_2 \cap Ƚ_3)$,
v) $(Ƚ_1 \cup Ƚ_2)^c = {Ƚ_1}^c \cap {Ƚ_2}^2$,
vi) $(Ƚ_1 \cap Ƚ_2)^c = {Ƚ_1}^c \cup {Ƚ_2}^2$.

Proof: The proofs of the given identities are demonstrated easily.

Definition 15: Let $Ƚ_1 = \begin{pmatrix} \left([s_{\mathbb{a}_1}, s_{\mathbb{b}_1}]e^{i[s_{\alpha_1}, s_{\beta_1}]}, [s_{\mathbb{c}_1}, s_{\mathbb{d}_1}]e^{i[s_{\gamma_1}, s_{\delta_1}]}\right), \\ \left(s_{\mathfrak{u}_1}e^{is_{Ֆ_1}}, s_{\mathfrak{v}_1}e^{is_{\mathfrak{n}_1}}\right) \end{pmatrix}$ and $Ƚ_2 = \begin{pmatrix} \left([s_{\mathbb{a}_2}, s_{\mathbb{b}_2}]e^{i[s_{\alpha_2}, s_{\beta_2}]}, [s_{\mathbb{c}_2}, s_{\mathbb{d}_2}]e^{i[s_{\gamma_2}, s_{\delta_2}]}\right), \\ \left(s_{\mathfrak{u}_2}e^{is_{Ֆ_2}}, s_{\mathfrak{v}_2}e^{is_{\mathfrak{n}_2}}\right) \end{pmatrix}$ be two LCCuPFNs over $\mathfrak{T}_{[0,\hbar]}$ then, $Ƚ_1 \preccurlyeq Ƚ_2$ if $\mathbb{a}_1 \leq \mathbb{a}_2, \mathbb{b}_1 \leq \mathbb{b}_2, \mathbb{c}_1 \geq$

$\mathbb{c}_2, \mathbb{d}_1 \geq \mathbb{d}_2, \alpha_1 \leq \alpha_2, \beta_1 \leq \beta_2, \gamma_1 \geq \gamma_2, \delta_1 \geq \delta_2, \mathfrak{u}_1 \leq \mathfrak{u}_2, \mathfrak{v}_1 \geq \mathfrak{v}_2, \mathfrak{D}_1 \leq \mathfrak{D}_2$, and $\mathfrak{n}_1 \geq \mathfrak{n}_2$.

If an order "≼" is satisfied for LCCuPFNs Ⱡ_1 and Ⱡ_2 then, it is said to be LCCuPF-admissible order. However, the LCCuPF-admissible order does not hold for all LCCuPFNs. Therefore, for comparison of two LCCuPFNs, we demonstrate score and accuracy functions which play a critical role in transforming a LCCuPFN to a comparable numerical value in the context of linguistic term set.

Definition 16: Suppose $\text{Ⱡ} = \begin{pmatrix} \left([s_{\mathbb{a}}, s_{\mathbb{b}}]e^{i[s_\alpha, s_\beta]}, [s_{\mathbb{c}}, s_{\mathbb{d}}]e^{i[s_\gamma, s_\delta]}\right), \\ \left(s_{\mathfrak{u}}e^{is_{\mathfrak{D}}}, s_{\mathfrak{v}}e^{is_{\mathfrak{n}}}\right) \end{pmatrix}$ be a LCCuPFN. Then, its score function (S.F) and accuracy function (A.F) are respectively demonstrated as:

$$S.F(\text{Ⱡ}) = s_{\sqrt{\frac{6\hbar^2 + \mathbb{a}^2 - \mathbb{c}^2 + \mathbb{b}^2 - \mathbb{d}^2 + \alpha^2 - \gamma^2 + \beta^2 - \delta^2 + \mathfrak{u}^2 - \mathfrak{v}^2 + \mathfrak{D}^2 - \mathfrak{n}^2}{12}}}.$$

$$A.F(\text{Ⱡ}) = s_{\sqrt{\frac{\mathbb{a}^2 + \mathbb{c}^2 + \mathbb{b}^2 + \mathbb{d}^2 + \alpha^2 + \gamma^2 + \beta^2 + \delta^2 + \mathfrak{u}^2 + \mathfrak{v}^2 + \mathfrak{D}^2 + \mathfrak{n}^2}{6}}}.$$

1. Basic Operational Laws (OLs) for Linguistic Complex Cubic Pythagorean Fuzzy Sets

Here, we establish some basic OLs for LCCuPFNs which play a critical role in defining AOs. In the proposed frameworks, the LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ and $\lambda > 0$. The proposed notions are demonstrated below.

Definition 17: Let $\text{Ⱡ}_1 = \begin{pmatrix} \left([s_{\mathbb{a}_1}, s_{\mathbb{b}_1}]e^{i[s_{\alpha_1}, s_{\beta_1}]}, [s_{\mathbb{c}_1}, s_{\mathbb{d}_1}]e^{i[s_{\gamma_1}, s_{\delta_1}]}\right), \\ \left(s_{\mathfrak{u}_1}e^{is_{\mathfrak{D}_1}}, s_{\mathfrak{v}_1}e^{is_{\mathfrak{n}_1}}\right) \end{pmatrix}$ and $\text{Ⱡ}_2 = \begin{pmatrix} \left([s_{\mathbb{a}_2}, s_{\mathbb{b}_2}]e^{i[s_{\alpha_2}, s_{\beta_2}]}, [s_{\mathbb{c}_2}, s_{\mathbb{d}_2}]e^{i[s_{\gamma_2}, s_{\delta_2}]}\right), \\ \left(s_{\mathfrak{u}_2}e^{is_{\mathfrak{D}_2}}, s_{\mathfrak{v}_2}e^{is_{\mathfrak{n}_2}}\right) \end{pmatrix}$ be two LCCuPFNs over $\mathbb{S}_{[0,\hbar]}$ and $\lambda > 0$. Then, the basic OLs for LCCuPFNs are established as:

1) $$\text{Ⱡ}_1 \oplus \text{Ⱡ}_2 = \left(\begin{pmatrix} \left[s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{a}_2^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2^2}{\hbar^2}\right)}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\alpha_1^2}{\hbar^2}\right)\left(1-\frac{\alpha_2^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{\beta_1^2}{\hbar^2}\right)\left(1-\frac{\beta_2^2}{\hbar^2}\right)}}\right]}, \\ \left[s_{\hbar\left(\frac{\mathbb{c}_1.\mathbb{c}_2}{\hbar^2}\right)}, s_{\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)}\right] e^{i\left[s_{\hbar\left(\frac{\gamma_1.\gamma_2}{\hbar^2}\right)}, s_{\hbar\left(\frac{\delta_1.\delta_2}{\hbar^2}\right)}\right]} \end{pmatrix}, \begin{pmatrix} s_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{u}_2^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{D}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{D}_2^2}{\hbar^2}\right)}}} \\ s_{\hbar\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hbar^2}\right)} e^{is_{\hbar\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hbar^2}\right)}} \end{pmatrix}\right).$$

2) $$\text{Ⱡ}_1 \otimes \text{Ⱡ}_2 = \left(\begin{pmatrix} \left[s_{\hbar\left(\frac{\mathbb{a}_1.\mathbb{a}_2}{\hbar^2}\right)}, s_{\hbar\left(\frac{\mathbb{b}_1.\mathbb{b}_2}{\hbar^2}\right)}\right] e^{i\left[s_{\hbar\left(\frac{\alpha_1.\alpha_2}{\hbar^2}\right)}, s_{\hbar\left(\frac{\beta_1.\beta_2}{\hbar^2}\right)}\right]}, \\ \left[s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{c}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{c}_2^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{d}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{d}_2^2}{\hbar^2}\right)}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\gamma_1^2}{\hbar^2}\right)\left(1-\frac{\gamma_2^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{\delta_1^2}{\hbar^2}\right)\left(1-\frac{\delta_2^2}{\hbar^2}\right)}}\right]} \end{pmatrix}, \begin{pmatrix} s_{\hbar\left(\frac{\mathfrak{u}_1.\mathfrak{u}_2}{\hbar^2}\right)} e^{is_{\hbar\left(\frac{\mathfrak{D}_1.\mathfrak{D}_2}{\hbar^2}\right)}} \\ s_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{v}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{v}_2^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{n}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{n}_2^2}{\hbar^2}\right)}}} \end{pmatrix}\right).$$

3) $\lambda\text{Ⱡ}_1 \oplus \lambda\text{Ⱡ}_2 =$

$$\left(\left(\begin{array}{c}\left[s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{b}_2^2}{\hbar^2}\right)^{\lambda}}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\alpha_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\alpha_2^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\beta_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\beta_2^2}{\hbar^2}\right)^{\lambda}}}\right]}, \\ \left[s_{\hbar\left(\frac{\mathbb{c}_1.\mathbb{c}_2}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^{\lambda}}\right] e^{i\left[s_{\hbar\left(\frac{\gamma_1.\gamma_2}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\delta_1.\delta_2}{\hbar^2}\right)^{\lambda}}\right]}\end{array}\right), \\ \left(\begin{array}{c} s_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{u}_2^2}{\hbar^2}\right)^{\lambda}}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{s}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{s}_2^2}{\hbar^2}\right)^{\lambda}}}} \\ s_{\hbar\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hbar^2}\right)^{\lambda}} e^{is_{\hbar\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hbar^2}\right)^{\lambda}}}\end{array}\right)\right).$$

4) $\lambda Ⱡ_1 \otimes \lambda Ⱡ_2 =$

$$\left(\left(\begin{array}{c}\left[s_{\hbar\left(\frac{\mathbb{a}_1.\mathbb{a}_2}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\mathbb{b}_1.\mathbb{b}_2}{\hbar^2}\right)^{\lambda}}\right] e^{i\left[s_{\hbar\left(\frac{\alpha_1.\alpha_2}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\beta_1.\beta_2}{\hbar^2}\right)^{\lambda}}\right]}, \\ \left[s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{c}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{c}_2^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{d}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{d}_2^2}{\hbar^2}\right)^{\lambda}}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\gamma_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\gamma_2^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\delta_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\delta_2^2}{\hbar^2}\right)^{\lambda}}}\right]}\end{array}\right), \\ \left(\begin{array}{c} s_{\hbar\left(\frac{\mathfrak{u}_1.\mathfrak{u}_2}{\hbar^2}\right)^{\lambda}} e^{is_{\hbar\left(\frac{\mathfrak{s}_1.\mathfrak{s}_2}{\hbar^2}\right)^{\lambda}}} \\ s_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{v}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{v}_2^2}{\hbar^2}\right)^{\lambda}}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{n}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{n}_2^2}{\hbar^2}\right)^{\lambda}}}}\end{array}\right)\right)$$

Theorem 2: Let $Ⱡ_1 = \left(\begin{array}{c}\left(\left[s_{\mathbb{a}_1}, s_{\mathbb{b}_1}\right]e^{i\left[s_{\alpha_1}, s_{\beta_1}\right]}, \left[s_{\mathbb{c}_1}, s_{\mathbb{d}_1}\right]e^{i\left[s_{\gamma_1}, s_{\delta_1}\right]}\right), \\ \left(s_{\mathfrak{u}_1}e^{is_{\mathfrak{s}_1}}, s_{\mathfrak{v}_1}e^{is_{\mathfrak{n}_1}}\right)\end{array}\right)$ and $Ⱡ_2 = \left(\begin{array}{c}\left(\left[s_{\mathbb{a}_2}, s_{\mathbb{b}_2}\right]e^{i\left[s_{\alpha_2}, s_{\beta_2}\right]}, \left[s_{\mathbb{c}_2}, s_{\mathbb{d}_2}\right]e^{i\left[s_{\gamma_2}, s_{\delta_2}\right]}\right), \\ \left(s_{\mathfrak{u}_2}e^{is_{\mathfrak{s}_2}}, s_{\mathfrak{v}_2}e^{is_{\mathfrak{n}_2}}\right)\end{array}\right)$ be two LCCuPFNs over $\mathbb{S}_{[0,\hbar]}$ and $\lambda > 0$. Then,

i) $Ⱡ_1 ⨸ Ⱡ_2$ is an LCCuPFN.
ii) $Ⱡ_1 \otimes Ⱡ_2$ is an LCCuPFN.
iii) $\lambda Ⱡ_1 ⨸ \lambda Ⱡ_2$ is an LCCuPFN.
iv) $\lambda Ⱡ_1 \otimes \lambda Ⱡ_2$ is an LCCuPFN.

Proof: i). Let $Ⱡ_1 = \left(\begin{array}{c}\left(\left[s_{\mathbb{a}_1}, s_{\mathbb{b}_1}\right]e^{i\left[s_{\alpha_1}, s_{\beta_1}\right]}, \left[s_{\mathbb{c}_1}, s_{\mathbb{d}_1}\right]e^{i\left[s_{\gamma_1}, s_{\delta_1}\right]}\right), \\ \left(s_{\mathfrak{u}_1}e^{is_{\mathfrak{s}_1}}, s_{\mathfrak{v}_1}e^{is_{\mathfrak{n}_1}}\right)\end{array}\right)$ and $Ⱡ_2 = \left(\begin{array}{c}\left(\left[s_{\mathbb{a}_2}, s_{\mathbb{b}_2}\right]e^{i\left[s_{\alpha_2}, s_{\beta_2}\right]}, \left[s_{\mathbb{c}_2}, s_{\mathbb{d}_2}\right]e^{i\left[s_{\gamma_2}, s_{\delta_2}\right]}\right), \\ \left(s_{\mathfrak{u}_2}e^{is_{\mathfrak{s}_2}}, s_{\mathfrak{v}_2}e^{is_{\mathfrak{n}_2}}\right)\end{array}\right)$ be two LCCuPFNs over $\mathbb{S}_{[0,\hbar]}$. By Definition of $Ⱡ_1 ⨸ Ⱡ_2$, we have

$$\text{Ł}_1 \text{⨸} \text{Ł}_2 = \left(\begin{matrix} \left(\begin{matrix} \left[\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathbb{a}_2{}^2}{\hbar^2}\right)}}, \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2{}^2}{\hbar^2}\right)}} \right] e^{i\left[\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\alpha_1{}^2}{\hbar^2}\right)\left(1-\frac{\alpha_2{}^2}{\hbar^2}\right)}}, \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\beta_1{}^2}{\hbar^2}\right)\left(1-\frac{\beta_2{}^2}{\hbar^2}\right)}}\right]}, \\ \left[\mathfrak{s}_{\hbar\left(\frac{\mathbb{c}_1.\mathbb{c}_2}{\hbar^2}\right)}, \mathfrak{s}_{\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)}\right] e^{i\left[\mathfrak{s}_{\hbar\left(\frac{\gamma_1.\gamma_2}{\hbar^2}\right)}, \mathfrak{s}_{\hbar\left(\frac{\delta_1.\delta_2}{\hbar^2}\right)}\right]} \end{matrix} \right), \\ \left(\begin{matrix} \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{u}_2{}^2}{\hbar^2}\right)}} e^{i\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{S}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{S}_2{}^2}{\hbar^2}\right)}}} \\ \mathfrak{s}_{\hbar\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hbar^2}\right)} e^{i\mathfrak{s}_{\hbar\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hbar^2}\right)}} \end{matrix} \right) \end{matrix} \right).$$

Now, we have to show that $\left(\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2{}^2}{\hbar^2}\right)}\right)^2 + \left(\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)\right)^2 \le \hbar^2$. Since,

$$\left(\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2{}^2}{\hbar^2}\right)}\right)^2 + \left(\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)\right)^2 = \hbar^2\left(1-\left(1-\frac{\mathbb{b}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2{}^2}{\hbar^2}\right)\right) + \hbar^2\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^2$$

$$= \hbar^2\left(\left(1-\left(1-\frac{\mathbb{b}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2{}^2}{\hbar^2}\right)\right) + \left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^2\right)$$

$$= \hbar^2\left(\left(1-\left(\frac{\hbar^2-\mathbb{b}_1{}^2}{\hbar^2}\right)\left(\frac{\hbar^2-\mathbb{b}_2{}^2}{\hbar^2}\right)\right) + \left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^2\right)$$

$$\le \hbar^2\left(\left(1-\left(\frac{\mathbb{d}_1{}^2}{\hbar^2}\right)\left(\frac{\mathbb{d}_2{}^2}{\hbar^2}\right)\right) + \left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^2\right) = \hbar^2\left(1-\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^2 + \left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^2\right) = \hbar^2.$$

Thus, $\left(\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2{}^2}{\hbar^2}\right)}\right)^2 + \left(\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)\right)^2 \le \hbar^2$.

Similarly, we can easily show the other conditions of LCCuPFSs, that is;

$$\left(\hbar\sqrt{1-\left(1-\frac{\beta_1{}^2}{\hbar^2}\right)\left(1-\frac{\beta_2{}^2}{\hbar^2}\right)}\right)^2 + \left(\hbar\left(\frac{\delta_1.\delta_2}{\hbar^2}\right)\right)^2 \le \hbar^2.$$

$$\left(\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{u}_2{}^2}{\hbar^2}\right)}\right)^2 + \left(\hbar\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hbar^2}\right)\right)^2 \le \hbar^2.$$

$$\left(\hbar\sqrt{1-\left(1-\frac{\mathfrak{S}_1{}^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{S}_2{}^2}{\hbar^2}\right)}\right)^2 + \left(\hbar\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hbar^2}\right)\right)^2 \le \hbar^2.$$

Thus, $\text{Ł}_1 \text{⨸} \text{Ł}_2$ is an LCCuPFN.

ii) similarly, we can prove (ii).

iii) Let $\text{Ł}_1 = \left(\begin{matrix} \left(\left[\mathfrak{s}_{\mathbb{a}_1}, \mathfrak{s}_{\mathbb{b}_1}\right] e^{i\left[\mathfrak{s}_{\alpha_1}, \mathfrak{s}_{\beta_1}\right]}, \left[\mathfrak{s}_{\mathbb{c}_1}, \mathfrak{s}_{\mathbb{d}_1}\right] e^{i\left[\mathfrak{s}_{\gamma_1}, \mathfrak{s}_{\delta_1}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_1} e^{i\mathfrak{s}_{\mathfrak{S}_1}}, \mathfrak{s}_{\mathfrak{v}_1} e^{i\mathfrak{s}_{\mathfrak{n}_1}}\right) \end{matrix} \right)$ and $\text{Ł}_2 = \left(\begin{matrix} \left(\left[\mathfrak{s}_{\mathbb{a}_2}, \mathfrak{s}_{\mathbb{b}_2}\right] e^{i\left[\mathfrak{s}_{\alpha_2}, \mathfrak{s}_{\beta_2}\right]}, \left[\mathfrak{s}_{\mathbb{c}_2}, \mathfrak{s}_{\mathbb{d}_2}\right] e^{i\left[\mathfrak{s}_{\gamma_2}, \mathfrak{s}_{\delta_2}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_2} e^{i\mathfrak{s}_{\mathfrak{S}_2}}, \mathfrak{s}_{\mathfrak{v}_2} e^{i\mathfrak{s}_{\mathfrak{n}_2}}\right) \end{matrix} \right)$

be two LCCuPFNs over $\mathbb{S}_{[0,\hslash]}$. By Definition of $\lambda Ⱡ_1 \odiv \lambda Ⱡ_2$, we have

$$\lambda Ⱡ_1 \odiv \lambda Ⱡ_2 = \left(\left(\begin{array}{c}\left[\mathfrak{s}_{\hslash\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hslash^2}\right)^{\lambda}}}, \mathfrak{s}_{\hslash\sqrt{1-\left(1-\frac{\mathbb{b}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathbb{b}_2^2}{\hslash^2}\right)^{\lambda}}}\right] e^{i\left[\mathfrak{s}_{\hslash\sqrt{1-\left(1-\frac{\alpha_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\alpha_2^2}{\hslash^2}\right)^{\lambda}}}, \mathfrak{s}_{\hslash\sqrt{1-\left(1-\frac{\beta_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\beta_2^2}{\hslash^2}\right)^{\lambda}}}\right]}, \\ \left[\mathfrak{s}_{\hslash\left(\frac{\mathbb{c}_1.\mathbb{c}_2}{\hslash^2}\right)^{\lambda}}, \mathfrak{s}_{\hslash\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}}\right] e^{i\left[\mathfrak{s}_{\hslash\left(\frac{\gamma_1.\gamma_2}{\hslash^2}\right)^{\lambda}}, \mathfrak{s}_{\hslash\left(\frac{\delta_1.\delta_2}{\hslash^2}\right)^{\lambda}}\right]}\end{array}\right), \left(\begin{array}{c}\mathfrak{s}_{\hslash\sqrt{1-\left(1-\frac{\mathfrak{u}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathfrak{u}_2^2}{\hslash^2}\right)^{\lambda}}} e^{i\mathfrak{s}_{\hslash\sqrt{1-\left(1-\frac{\mathfrak{d}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathfrak{d}_2^2}{\hslash^2}\right)^{\lambda}}}} \\ \mathfrak{s}_{\hslash\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hslash^2}\right)^{\lambda}} e^{i\mathfrak{s}_{\hslash\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hslash^2}\right)^{\lambda}}}\end{array}\right)\right)$$

We have to show that $\left(\hslash\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hslash^2}\right)^{\lambda}}\right)^2 + \left(\hslash\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2 \le \hslash^2$. Now,

$$\left(\hslash\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hslash^2}\right)^{\lambda}}\right)^2 + \left(\hslash\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2 = \hslash^2\left(1-\left(1-\frac{\mathbb{a}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hslash^2}\right)^{\lambda}\right) + \hslash^2\left(\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2$$

$$= \hslash^2\left(\left(1-\left(1-\frac{\mathbb{a}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hslash^2}\right)^{\lambda}\right) + \left(\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2\right)$$

$$= \hslash^2\left(\left(1-\left(\frac{\hslash^2-\mathbb{a}_1^2}{\hslash^2}\right)^{\lambda}\left(\frac{\hslash^2-\mathbb{a}_2^2}{\hslash^2}\right)^{\lambda}\right) + \left(\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2\right)$$

$$\le \hslash^2\left(\left(1-\left(\frac{\mathbb{d}_1^2}{\hslash^2}\right)^{\lambda}\left(\frac{\mathbb{d}_2^2}{\hslash^2}\right)^{\lambda}\right) + \left(\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2\right) = \hslash^2\left(1-\left(\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2 + \left(\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2\right)$$

$$= \hslash^2.$$

Thus, $\left(\hslash\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hslash^2}\right)^{\lambda}}\right)^2 + \left(\hslash\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hslash^2}\right)^{\lambda}\right)^2 \le \hslash^2$.

Similarly, we can prove the other inequalities, that is;

$$\left(\hslash\sqrt{1-\left(1-\frac{\beta_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\beta_2^2}{\hslash^2}\right)^{\lambda}}\right)^2 + \left(\hslash\left(\frac{\delta_1.\delta_2}{\hslash^2}\right)^{\lambda}\right)^2 \le \hslash^2.$$

$$\left(\hslash\sqrt{1-\left(1-\frac{\mathfrak{u}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathfrak{u}_2^2}{\hslash^2}\right)^{\lambda}}\right)^2 + \left(\hslash\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hslash^2}\right)^{\lambda}\right)^2 \le \hslash^2.$$

$$\left(\hslash\sqrt{1-\left(1-\frac{\mathfrak{d}_1^2}{\hslash^2}\right)^{\lambda}\left(1-\frac{\mathfrak{d}_2^2}{\hslash^2}\right)^{\lambda}}\right)^2 + \left(\hslash\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hslash^2}\right)^{\lambda}\right)^2 \le \hslash^2.$$

Thus, $\lambda Ⱡ_1 \odiv \lambda Ⱡ_2$ is a LCCuPFN.

iv) Similarly, we can prove (iv).

Theorem 3: Let $\text{Ⱡ}_1 = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_1}, \mathfrak{s}_{\mathbb{b}_1}\right] e^{i\left[\mathfrak{s}_{\alpha_1}, \mathfrak{s}_{\beta_1}\right]}, \left[\mathfrak{s}_{\mathbb{c}_1}, \mathfrak{s}_{\mathbb{d}_1}\right] e^{i\left[\mathfrak{s}_{\gamma_1}, \mathfrak{s}_{\delta_1}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_1} e^{i\mathfrak{s}_{\mathfrak{D}_1}}, \mathfrak{s}_{\mathfrak{v}_1} e^{i\mathfrak{s}_{\mathfrak{n}_1}}\right) \end{pmatrix}$ and $\text{Ⱡ}_2 = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_2}, \mathfrak{s}_{\mathbb{b}_2}\right] e^{i\left[\mathfrak{s}_{\alpha_2}, \mathfrak{s}_{\beta_2}\right]}, \left[\mathfrak{s}_{\mathbb{c}_2}, \mathfrak{s}_{\mathbb{d}_2}\right] e^{i\left[\mathfrak{s}_{\gamma_2}, \mathfrak{s}_{\delta_2}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_2} e^{i\mathfrak{s}_{\mathfrak{D}_2}}, \mathfrak{s}_{\mathfrak{v}_2} e^{i\mathfrak{s}_{\mathfrak{n}_2}}\right) \end{pmatrix}$ be two LCCuPFNs over $\mathbb{S}_{[0,\hbar]}$ and $\lambda > 0$. Then,

i) $\text{Ⱡ}_1 \oplus \text{Ⱡ}_2 = \text{Ⱡ}_2 \oplus \text{Ⱡ}_2$.

ii) $\text{Ⱡ}_1 \otimes \text{Ⱡ}_2 = \text{Ⱡ}_2 \otimes \text{Ⱡ}_1$.

iii) $\lambda\text{Ⱡ}_1 \oplus \lambda\text{Ⱡ}_2 = \lambda\text{Ⱡ}_2 \oplus \lambda\text{Ⱡ}_1$.

iv) $\lambda\text{Ⱡ}_1 \otimes \lambda\text{Ⱡ}_2 = \lambda\text{Ⱡ}_2 \otimes \lambda\text{Ⱡ}_1$.

Proof: Let $\text{Ⱡ}_1 = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_1}, \mathfrak{s}_{\mathbb{b}_1}\right] e^{i\left[\mathfrak{s}_{\alpha_1}, \mathfrak{s}_{\beta_1}\right]}, \left[\mathfrak{s}_{\mathbb{c}_1}, \mathfrak{s}_{\mathbb{d}_1}\right] e^{i\left[\mathfrak{s}_{\gamma_1}, \mathfrak{s}_{\delta_1}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_1} e^{i\mathfrak{s}_{\mathfrak{D}_1}}, \mathfrak{s}_{\mathfrak{v}_1} e^{i\mathfrak{s}_{\mathfrak{n}_1}}\right) \end{pmatrix}$ and $\text{Ⱡ}_2 = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_2}, \mathfrak{s}_{\mathbb{b}_2}\right] e^{i\left[\mathfrak{s}_{\alpha_2}, \mathfrak{s}_{\beta_2}\right]}, \left[\mathfrak{s}_{\mathbb{c}_2}, \mathfrak{s}_{\mathbb{d}_2}\right] e^{i\left[\mathfrak{s}_{\gamma_2}, \mathfrak{s}_{\delta_2}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_2} e^{i\mathfrak{s}_{\mathfrak{D}_2}}, \mathfrak{s}_{\mathfrak{v}_2} e^{i\mathfrak{s}_{\mathfrak{n}_2}}\right) \end{pmatrix}$ be two LCCuPFNs over $\mathfrak{T}_{[0,\hbar]}$ and $\lambda > 0$. Then,

i). By Definition of $\text{Ⱡ}_1 \oplus \text{Ⱡ}_2$, we have

$$\text{Ⱡ}_1 \oplus \text{Ⱡ}_2 = \begin{pmatrix} \begin{pmatrix} \left[\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{a}_2^2}{\hbar^2}\right)}}, \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2^2}{\hbar^2}\right)}}\right] e^{i\left[\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\alpha_1^2}{\hbar^2}\right)\left(1-\frac{\alpha_2^2}{\hbar^2}\right)}}, \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\beta_1^2}{\hbar^2}\right)\left(1-\frac{\beta_2^2}{\hbar^2}\right)}}\right]}, \\ \left[\mathfrak{s}_{\hbar\left(\frac{\mathbb{c}_1 \cdot \mathbb{c}_2}{\hbar^2}\right)}, \mathfrak{s}_{\hbar\left(\frac{\mathbb{d}_1 \cdot \mathbb{d}_2}{\hbar^2}\right)}\right] e^{i\left[\mathfrak{s}_{\hbar\left(\frac{\gamma_1 \cdot \gamma_2}{\hbar^2}\right)}, \mathfrak{s}_{\hbar\left(\frac{\delta_1 \cdot \delta_2}{\hbar^2}\right)}\right]} \end{pmatrix}, \\ \begin{pmatrix} \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{u}_2^2}{\hbar^2}\right)}} e^{i\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{D}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{D}_2^2}{\hbar^2}\right)}}} \\ \mathfrak{s}_{\hbar\left(\frac{\mathfrak{v}_1 \cdot \mathfrak{v}_2}{\hbar^2}\right)} e^{i\mathfrak{s}_{\hbar\left(\frac{\mathfrak{n}_1 \cdot \mathfrak{n}_2}{\hbar^2}\right)}} \end{pmatrix} \end{pmatrix} =$$

$$\begin{pmatrix} \begin{pmatrix} \left[\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_2^2}{\hbar^2}\right)\left(1-\frac{\mathbb{a}_1^2}{\hbar^2}\right)}}, \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_2^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_1^2}{\hbar^2}\right)}}\right] e^{i\left[\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\alpha_2^2}{\hbar^2}\right)\left(1-\frac{\alpha_1^2}{\hbar^2}\right)}}, \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\beta_2^2}{\hbar^2}\right)\left(1-\frac{\beta_1^2}{\hbar^2}\right)}}\right]}, \\ \left[\mathfrak{s}_{\hbar\left(\frac{\mathbb{c}_2 \cdot \mathbb{c}_1}{\hbar^2}\right)}, \mathfrak{s}_{\hbar\left(\frac{\mathbb{d}_2 \cdot \mathbb{d}_1}{\hbar^2}\right)}\right] e^{i\left[\mathfrak{s}_{\hbar\left(\frac{\gamma_2 \cdot \gamma_1}{\hbar^2}\right)}, \mathfrak{s}_{\hbar\left(\frac{\delta_2 \cdot \delta_1}{\hbar^2}\right)}\right]} \end{pmatrix}, \\ \begin{pmatrix} \mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_2^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{u}_1^2}{\hbar^2}\right)}} e^{i\mathfrak{s}_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{D}_2^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{D}_1^2}{\hbar^2}\right)}}} \\ \mathfrak{s}_{\hbar\left(\frac{\mathfrak{v}_2 \cdot \mathfrak{v}_1}{\hbar^2}\right)} e^{i\mathfrak{s}_{\hbar\left(\frac{\mathfrak{n}_2 \cdot \mathfrak{n}_1}{\hbar^2}\right)}} \end{pmatrix} \end{pmatrix}$$

$$\text{Ⱡ}_1 \oplus \text{Ⱡ}_2 = \text{Ⱡ}_2 \oplus \text{Ⱡ}_1.$$

ii) It can be easily proven.

iii) By Definition of $\lambda\text{Ⱡ}_1 \oplus \lambda\text{Ⱡ}_2$, we have

$$\lambda Ł_1 \oplus \lambda Ł_2$$

$$= \left(\begin{array}{c}\left(\left[s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_2^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{b}_2^2}{\hbar^2}\right)^{\lambda}}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\alpha_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\alpha_2^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\beta_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\beta_2^2}{\hbar^2}\right)^{\lambda}}}\right]}, \atop \left[s_{\hbar\left(\frac{\mathbb{c}_1.\mathbb{c}_2}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)^{\lambda}}\right] e^{i\left[s_{\hbar\left(\frac{\gamma_1.\gamma_2}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\delta_1.\delta_2}{\hbar^2}\right)^{\lambda}}\right]}\right), \\ \left(\begin{array}{c} s_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{u}_2^2}{\hbar^2}\right)^{\lambda}}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{d}_1^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{d}_2^2}{\hbar^2}\right)^{\lambda}}}} \\ s_{\hbar\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hbar^2}\right)^{\lambda}} e^{is_{\hbar\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hbar^2}\right)^{\lambda}}}\end{array}\right)\end{array}\right)$$

$$= \left(\begin{array}{c}\left(\left[s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_2^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{a}_1^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_2^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathbb{b}_1^2}{\hbar^2}\right)^{\lambda}}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\alpha_2^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\alpha_1^2}{\hbar^2}\right)^{\lambda}}}, s_{\hbar\sqrt{1-\left(1-\frac{\beta_2^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\beta_1^2}{\hbar^2}\right)^{\lambda}}}\right]}, \atop \left[s_{\hbar\left(\frac{\mathbb{c}_2.\mathbb{c}_1}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\mathbb{d}_2.\mathbb{d}_1}{\hbar^2}\right)^{\lambda}}\right] e^{i\left[s_{\hbar\left(\frac{\gamma_2.\gamma_1}{\hbar^2}\right)^{\lambda}}, s_{\hbar\left(\frac{\delta_2.\delta_1}{\hbar^2}\right)^{\lambda}}\right]}\right), \\ \left(\begin{array}{c} s_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_2^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{u}_1^2}{\hbar^2}\right)^{\lambda}}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{d}_2^2}{\hbar^2}\right)^{\lambda}\left(1-\frac{\mathfrak{d}_1^2}{\hbar^2}\right)^{\lambda}}}} \\ s_{\hbar\left(\frac{\mathfrak{v}_2.\mathfrak{v}_1}{\hbar^2}\right)^{\lambda}} e^{is_{\hbar\left(\frac{\mathfrak{n}_2.\mathfrak{n}_1}{\hbar^2}\right)^{\lambda}}}\end{array}\right)\end{array}\right).$$

Thus, $\lambda Ł_1 \oplus \lambda Ł_2 = \lambda Ł_2 \oplus \lambda Ł_1$.

iv) It can be proven similarly.

Theorem 4: Let $Ł_1 = \left(\begin{array}{c}\left(\left[s_{\mathbb{a}_1}, s_{\mathbb{b}_1}\right]e^{i\left[s_{\alpha_1}, s_{\beta_1}\right]}, \left[s_{\mathbb{c}_1}, s_{\mathbb{d}_1}\right]e^{i\left[s_{\gamma_1}, s_{\delta_1}\right]}\right), \\ \left(s_{\mathfrak{u}_1}e^{is_{\mathfrak{d}_1}}, s_{\mathfrak{v}_1}e^{is_{\mathfrak{n}_1}}\right)\end{array}\right)$, $Ł_2 = \left(\begin{array}{c}\left(\left[s_{\mathbb{a}_2}, s_{\mathbb{b}_2}\right]e^{i\left[s_{\alpha_2}, s_{\beta_2}\right]}, \left[s_{\mathbb{c}_2}, s_{\mathbb{d}_2}\right]e^{i\left[s_{\gamma_2}, s_{\delta_2}\right]}\right), \\ \left(s_{\mathfrak{u}_2}e^{is_{\mathfrak{d}_2}}, s_{\mathfrak{v}_2}e^{is_{\mathfrak{n}_2}}\right)\end{array}\right)$ and $Ł_3 = \left(\begin{array}{c}\left(\left[s_{\mathbb{a}_3}, s_{\mathbb{b}_3}\right]e^{i\left[s_{\alpha_3}, s_{\beta_3}\right]}, \left[s_{\mathbb{c}_3}, s_{\mathbb{d}_3}\right]e^{i\left[s_{\gamma_3}, s_{\delta_3}\right]}\right), \\ \left(s_{\mathfrak{u}_3}e^{is_{\mathfrak{d}_3}}, s_{\mathfrak{v}_3}e^{is_{\mathfrak{n}_3}}\right)\end{array}\right)$ be three LCCuPFNs over $\mathbb{S}_{[0,\hbar]}$ and $\lambda > 0$. Then,

i) $(Ł_1 \oplus Ł_2) \oplus Ł_3 = Ł_1 \oplus (Ł_2 \oplus Ł_3)$.
ii) $(Ł_1 \otimes Ł_2) \otimes Ł_3 = Ł_1 \otimes (Ł_2 \otimes Ł_3)$.
iii) $(\lambda Ł_1 \oplus \lambda Ł_2) \oplus \lambda Ł_3 = \lambda Ł_1 \oplus (\lambda Ł_2 \oplus \lambda Ł_3)$.
iv) $(\lambda Ł_1 \otimes \lambda Ł_2) \otimes \lambda Ł_3 = \lambda Ł_1 \otimes (\lambda Ł_2 \otimes \lambda Ł_3)$.

Proof: The proof is straightforward.

4. Linguistic Complex Cubic Pythagorean Fuzzy Aggregation Operators

This section aims to establish some new AOs such as averaging aggregation, geometric aggregation, weighted averaging aggregation and weighted geometric aggregation operators within the framework of LCCuPFSs. The

LCCuPFS framework presents a comprehensive and robust structure capturing multi-dimensional uncertainty by integrating complex-valued, interval-based, and linguistic information within a single framework. Suppose $Ƚ_i = \begin{pmatrix} \left(\left[s_{\mathbb{a}_i}, s_{\mathbb{b}_i}\right]e^{i\left[s_{\alpha_i}, s_{\beta_i}\right]}, \left[s_{\mathbb{c}_i}, s_{\mathbb{d}_i}\right]e^{i\left[s_{\gamma_i}, s_{\delta_i}\right]}\right), \\ \left(s_{\mathfrak{u}_i}e^{is_{\vartheta_i}}, s_{\mathfrak{v}_i}e^{is_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of LCCuPFNs and ω_i denotes positive weight vector such that $\sum_{i=1}^{n}\omega_i = 1$.

The newly defined AOs within the framework of LCCuPFSs are demonstrated below:

Definition 18: Suppose $Ƚ_i = \begin{pmatrix} \left(\left[s_{\mathbb{a}_i}, s_{\mathbb{b}_i}\right]e^{i\left[s_{\alpha_i}, s_{\beta_i}\right]}, \left[s_{\mathbb{c}_i}, s_{\mathbb{d}_i}\right]e^{i\left[s_{\gamma_i}, s_{\delta_i}\right]}\right), \\ \left(s_{\mathfrak{u}_i}e^{is_{\vartheta_i}}, s_{\mathfrak{v}_i}e^{is_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$. Then, the linguistic complex cubic Pythagorean fuzzy averaging aggregation operator (LCCuPFAAO) is a map LCCuPFAAO: $Ƚ^n \to Ƚ$ defined by

$$LCCuPFAAO(Ƚ_i, Ƚ_2, Ƚ_3, \dots, Ƚ_n) = \ominus_{i=1}{}^{n}\, Ƚ_i,$$

$$LCCuPFAAO(Ƚ_i, Ƚ_2, Ƚ_3, \dots, Ƚ_n) = \begin{pmatrix} \begin{pmatrix} \left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{a}_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{b}_i^2}{\hbar^2}\right)}}\right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\beta_i^2}{\hbar^2}\right)}}\right]}, \\ \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{c}_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{d}_i}{\hbar}\right)\right)}\right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\delta_i}{\hbar}\right)\right)}\right]} \end{pmatrix}, \\ \begin{pmatrix} s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\vartheta_1^2}{\hbar^2}\right)}}} \\ s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{v}_i}{\hbar}\right)\right)} e^{is_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{n}_i}{\hbar}\right)\right)}} \end{pmatrix} \end{pmatrix}.$$

Definition 19: Suppose $Ƚ_i = \begin{pmatrix} \left(\left[s_{\mathbb{a}_i}, s_{\mathbb{b}_i}\right]e^{i\left[s_{\alpha_i}, s_{\beta_i}\right]}, \left[s_{\mathbb{c}_i}, s_{\mathbb{d}_i}\right]e^{i\left[s_{\gamma_i}, s_{\delta_i}\right]}\right), \\ \left(s_{\mathfrak{u}_i}e^{is_{\vartheta_i}}, s_{\mathfrak{v}_i}e^{is_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$. Then, the linguistic complex cubic Pythagorean fuzzy geometric aggregation operator (LCCuPFGAO) is a map LCCuPFGAO: $Ƚ^n \to Ƚ$ defined by

$$LCCuPFGAO(Ƚ_i, Ƚ_2, Ƚ_3, \dots, Ƚ_n) = \otimes_{i=1}{}^{n}\, Ƚ_i,$$

$$LCCuPFAAO(Ƚ_i, Ƚ_2, Ƚ_3, \dots, Ƚ_n) = \begin{pmatrix} \begin{pmatrix} \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{a}_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{b}_i}{\hbar}\right)\right)}\right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\alpha_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\beta_i}{\hbar}\right)\right)}\right]}, \\ \left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{c}_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{d}_i^2}{\hbar^2}\right)}}\right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\gamma_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\delta_i^2}{\hbar^2}\right)}}\right]} \end{pmatrix}, \\ \begin{pmatrix} s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{u}_i}{\hbar}\right)\right)} e^{is_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\vartheta_i}{\hbar}\right)\right)}} \\ s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{v}_i^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{n}_i^2}{\hbar^2}\right)}}} \end{pmatrix} \end{pmatrix}.$$

Definition 20: Suppose $Ⱶ_i = \begin{pmatrix} \left([\mathrm{s}_{\mathbb{a}_i}, \mathrm{s}_{\mathbb{b}_i}]e^{i[s_{\alpha_i}, s_{\beta_i}]}, [\mathrm{s}_{\mathbb{c}_i}, \mathrm{s}_{\mathbb{d}_i}]e^{i[s_{\gamma_i}, s_{\delta_i}]}\right), \\ \left(\mathrm{s}_{\mathfrak{u}_i}e^{is_{\mathfrak{z}_i}}, \mathrm{s}_{\mathfrak{v}_i}e^{is_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ and $\omega_i > 0$ such that $\sum_{i=1}^{n} \omega_i = 1$. Then, the linguistic complex cubic Pythagorean fuzzy weighted averaging aggregation operator (LCCuPFWAAO) is a map LCCuPFWAAO: $Ⱶ^n \to Ⱶ$ defined by

$$LCCuPFWAAO(Ⱶ_i, Ⱶ_2, Ⱶ_3, \dots, Ⱶ_n) = \oplus_{i=1}{}^{n}\, \omega_i Ⱶ_i,$$

$$LCCuPFWAAO(Ⱶ_i, Ⱶ_2, Ⱶ_3, \dots, Ⱶ_n)$$

$$= \begin{pmatrix} \begin{pmatrix} \left[\mathrm{s}_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{a}_i^2}{\hbar^2}\right)^{\omega_i}}}, \mathrm{s}_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{b}_i^2}{\hbar^2}\right)^{\omega_i}}}\right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\beta_i^2}{\hbar^2}\right)^{\omega_i}}}\right]}, \\ \left[\mathrm{s}_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{c}_i}{\hbar}\right)^{\omega_i}\right)}, \mathrm{s}_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{d}_i}{\hbar}\right)^{\omega_i}\right)}\right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\delta_i}{\hbar}\right)^{\omega_i}\right)}\right]} \end{pmatrix}, \\ \begin{pmatrix} \mathrm{s}_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)^{\omega_i}}} e^{is_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{z}_1^2}{\hbar^2}\right)^{\omega_i}}}} \\ \mathrm{s}_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{v}_i}{\hbar}\right)^{\omega_i}\right)} e^{is_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{n}_i}{\hbar}\right)^{\omega_i}\right)}} \end{pmatrix} \end{pmatrix}.$$

Definition 21: Suppose $Ⱶ_i = \begin{pmatrix} \left([\mathrm{s}_{\mathbb{a}_i}, \mathrm{s}_{\mathbb{b}_i}]e^{i[s_{\alpha_i}, s_{\beta_i}]}, [\mathrm{s}_{\mathbb{c}_i}, \mathrm{s}_{\mathbb{d}_i}]e^{i[s_{\gamma_i}, s_{\delta_i}]}\right), \\ \left(\mathrm{s}_{\mathfrak{u}_i}e^{is_{\mathfrak{z}_i}}, \mathrm{s}_{\mathfrak{v}_i}e^{is_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ and $\omega_i > 0$ such that $\sum_{i=1}^{n} \omega_i = 1$. Then, the linguistic complex cubic Pythagorean fuzzy weighted geometric aggregation operator (LCCuPFWGAO) is a map LCCuPFWGAO: $Ⱶ^n \to Ⱶ$ defined by

$$LCCuPFWGAO(Ⱶ_i, Ⱶ_2, Ⱶ_3, \dots, Ⱶ_n) = \oplus_{i=1}{}^{n}\, \omega_i Ⱶ_i,$$

$$LCCuPFWGAO(Ⱶ_i, Ⱶ_2, Ⱶ_3, \dots, Ⱶ_n) = \begin{pmatrix} \begin{pmatrix} \left[\mathrm{s}_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{a}_i}{\hbar}\right)^{\omega_i}\right)}, \mathrm{s}_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{b}_i}{\hbar}\right)^{\omega_i}\right)}\right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\alpha_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\beta_i}{\hbar}\right)^{\omega_i}\right)}\right]}, \\ \left[\mathrm{s}_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{c}_i^2}{\hbar^2}\right)^{\omega_i}}}, \mathrm{s}_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{d}_i^2}{\hbar^2}\right)^{\omega_i}}}\right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\gamma_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\delta_i^2}{\hbar^2}\right)^{\omega_i}}}\right]} \end{pmatrix}, \\ \begin{pmatrix} \mathrm{s}_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{u}_i}{\hbar}\right)^{\omega_i}\right)} e^{is_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{z}_i}{\hbar}\right)^{\omega_i}\right)}} \\ \mathrm{s}_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{v}_i^2}{\hbar^2}\right)^{\omega_i}}} e^{is_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{n}_1^2}{\hbar^2}\right)^{\omega_i}}}} \end{pmatrix} \end{pmatrix}.$$

Theorem 5: Suppose $Ⱶ_i = \begin{pmatrix} \left([\mathrm{s}_{\mathbb{a}_i}, \mathrm{s}_{\mathbb{b}_i}]e^{i[s_{\alpha_i}, s_{\beta_i}]}, [\mathrm{s}_{\mathbb{c}_i}, \mathrm{s}_{\mathbb{d}_i}]e^{i[s_{\gamma_i}, s_{\delta_i}]}\right), \\ \left(\mathrm{s}_{\mathfrak{u}_i}e^{is_{\mathfrak{z}_i}}, \mathrm{s}_{\mathfrak{v}_i}e^{is_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ and $\omega_i > 0$ such that $\sum_{i=1}^{n} \omega_i = 1$. Then,

i) $LCCuPFAAO(\text{Ł}_i, \text{Ł}_2, \text{Ł}_3, \dots, \text{Ł}_n)$ is an LCCuPFN.

ii) $LCCuPFGAO(\text{Ł}_i, \text{Ł}_2, \text{Ł}_3, \dots, \text{Ł}_n)$ is an LCCuPFN.

iii) $LCCuPFWAAO(\text{Ł}_i, \text{Ł}_2, \text{Ł}_3, \dots, \text{Ł}_n)$ is an LCCuPFN.

iv) $LCCuPFWGAO(\text{Ł}_i, \text{Ł}_2, \text{Ł}_3, \dots, \text{Ł}_n)$ is an LCCuPFN.

Proof: i). We prove (i) using principle of mathematical induction (PMI). We first verify the basic condition, that is, for $n = 2$. Based on the OLs for LCCuPFNs as demonstrated in Theorem 2, we can directly prove the basic condition. Thus, by Definition $\text{Ł}_1 \oplus \text{Ł}_2$, we have

$$LCCuPFAAO(\text{Ł}_1, \text{Ł}_2) = \left(\begin{matrix} \left(\begin{matrix} \left[s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{a}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{a}_2^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{\mathbb{b}_1^2}{\hbar^2}\right)\left(1-\frac{\mathbb{b}_2^2}{\hbar^2}\right)}} \right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\alpha_1^2}{\hbar^2}\right)\left(1-\frac{\alpha_2^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{\beta_1^2}{\hbar^2}\right)\left(1-\frac{\beta_2^2}{\hbar^2}\right)}} \right]}, \\ \left[s_{\hbar\left(\frac{\mathbb{c}_1.\mathbb{c}_2}{\hbar^2}\right)}, s_{\hbar\left(\frac{\mathbb{d}_1.\mathbb{d}_2}{\hbar^2}\right)} \right] e^{i\left[s_{\hbar\left(\frac{\gamma_1.\gamma_2}{\hbar^2}\right)}, s_{\hbar\left(\frac{\delta_1.\delta_2}{\hbar^2}\right)} \right]} \end{matrix} \right), \\ \left(\begin{matrix} s_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{u}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{u}_2^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{D}_1^2}{\hbar^2}\right)\left(1-\frac{\mathfrak{D}_2^2}{\hbar^2}\right)}}} \\ s_{\hbar\left(\frac{\mathfrak{v}_1.\mathfrak{v}_2}{\hbar^2}\right)} e^{is_{\hbar\left(\frac{\mathfrak{n}_1.\mathfrak{n}_2}{\hbar^2}\right)}} \end{matrix} \right) \end{matrix} \right).$$

Hence, it is true for $n = 2$.

Suppose it is true for $n = k$. Now, we will verify for $n = k + 1$.

Now, for $n = k + 1$, we have

$$LCCuPFAAO(\text{Ł}_i, \text{Ł}_2, \text{Ł}_3, \dots, \text{Ł}_{k+1}) = LCCuPFAAO(\text{Ł}_i, \text{Ł}_2, \text{Ł}_3, \dots, \text{Ł}_k) \oplus \text{Ł}_{k+1}$$

$$= \left(\begin{matrix} \left(\begin{matrix} \left[s_{\hbar\sqrt{1-\prod_{i=1}^{k}\left(1-\frac{\mathbb{a}_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{k}\left(1-\frac{\mathbb{b}_i^2}{\hbar^2}\right)}} \right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{k}\left(1-\frac{\alpha_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{k}\left(1-\frac{\beta_i^2}{\hbar^2}\right)}} \right]}, \\ \left[s_{\hbar\left(\prod_{i=1}^{k}\left(\frac{\mathbb{c}_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{k}\left(\frac{\mathbb{d}_i}{\hbar}\right)\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{k}\left(\frac{\gamma_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{k}\left(\frac{\delta_i}{\hbar}\right)\right)} \right]} \end{matrix} \right), \\ \left(\begin{matrix} s_{\hbar\sqrt{1-\prod_{i=1}^{k}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\prod_{i=1}^{k}\left(1-\frac{\mathfrak{D}_1^2}{\hbar^2}\right)}}} \\ s_{\hbar\left(\prod_{i=1}^{k}\left(\frac{\mathfrak{v}_i}{\hbar}\right)\right)} e^{is_{\hbar\left(\prod_{i=1}^{k}\left(\frac{\mathfrak{n}_i}{\hbar}\right)\right)}} \end{matrix} \right) \end{matrix} \right) \oplus \text{Ł}_i$$

$$= \left(\begin{matrix} \left(\left[s_{\mathbb{a}_{k+1}}, s_{\mathbb{b}_{k+1}} \right] e^{i\left[s_{\alpha_{k+1}}, s_{\beta_{k+1}} \right]}, \left[s_{\mathbb{c}_{k+1}}, s_{\mathbb{d}_{k+1}} \right] e^{i\left[s_{\gamma_{k+1}}, s_{\delta_{k+1}} \right]} \right), \\ \left(s_{\mathfrak{u}_{k+1}} e^{is_{\mathfrak{D}_{k+1}}}, s_{\mathfrak{v}_{k+1}} e^{is_{\mathfrak{n}_{k+1}}} \right) \end{matrix} \right)$$

$$= \left(\begin{matrix} \left(\begin{matrix} \left[s_{\hbar\sqrt{1-\prod_{i=1}^{k+1}\left(1-\frac{\mathbb{a}_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{k+1}\left(1-\frac{\mathbb{b}_i^2}{\hbar^2}\right)}} \right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{k+1}\left(1-\frac{\alpha_i^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\prod_{i=1}^{k+1}\left(1-\frac{\beta_i^2}{\hbar^2}\right)}} \right]}, \\ \left[s_{\hbar\left(\prod_{i=1}^{k+1}\left(\frac{\mathbb{c}_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{k+1}\left(\frac{\mathbb{d}_i}{\hbar}\right)\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{k+1}\left(\frac{\gamma_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{k+1}\left(\frac{\delta_i}{\hbar}\right)\right)} \right]} \end{matrix} \right), \\ \left(\begin{matrix} s_{\hbar\sqrt{1-\prod_{i=1}^{k+1}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\prod_{i=1}^{k+1}\left(1-\frac{\mathfrak{D}_1^2}{\hbar^2}\right)}}} \\ s_{\hbar\left(\prod_{i=1}^{k+1}\left(\frac{\mathfrak{v}_i}{\hbar}\right)\right)} e^{is_{\hbar\left(\prod_{i=1}^{k+1}\left(\frac{\mathfrak{n}_i}{\hbar}\right)\right)}} \end{matrix} \right) \end{matrix} \right)$$

which holds for $n = k + 1$. Hence by PMI, $LCCuPFAAO(Ⱡ_i, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n)$ is an LCCuPFN.

Similarly, we can prove the other parts.

Theorem 6: Suppose $Ⱡ_i = \begin{pmatrix} \left(\left[s_{\mathbb{a}_i}, s_{\mathbb{b}_i}\right] e^{i\left[s_{\alpha_i}, s_{\beta_i}\right]}, \left[s_{\mathbb{c}_i}, s_{\mathbb{d}_i}\right] e^{i\left[s_{\gamma_i}, s_{\delta_i}\right]}\right), \\ \left(s_{\mathfrak{u}_i} e^{i s_{ϡ_i}}, s_{\mathfrak{v}_i} e^{i s_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ and $\omega_i > 0$ such that $Ⱡ_i = Ⱡ$, $\sum_{i=1}^{n} \omega_i = 1$. Then,

i) $LCCuPFWAAO(Ⱡ_i, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n) = Ⱡ$.

ii) $LCCuPFWGAO(Ⱡ_i, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n) = Ⱡ$.

Proof: i). Suppose $Ⱡ_i = \begin{pmatrix} \left(\left[s_{\mathbb{a}_i}, s_{\mathbb{b}_i}\right] e^{i\left[s_{\alpha_i}, s_{\beta_i}\right]}, \left[s_{\mathbb{c}_i}, s_{\mathbb{d}_i}\right] e^{i\left[s_{\gamma_i}, s_{\delta_i}\right]}\right), \\ \left(s_{\mathfrak{u}_i} e^{i s_{ϡ_i}}, s_{\mathfrak{v}_i} e^{i s_{\mathfrak{n}_i}}\right) \end{pmatrix}$ represents a collection of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ and $\omega_i > 0$ such that $Ⱡ_i = Ⱡ$, $\sum_{i=1}^{n} \omega_i = 1$. Then, by Definition of LCCuPFWAO, we have

$$LCCuPFWAAO(Ⱡ_1, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n)$$

$$= \left(\begin{pmatrix} \left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{a}_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{b}_i^2}{\hbar^2}\right)^{\omega_i}}} \right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\beta_i^2}{\hbar^2}\right)^{\omega_i}}} \right]}, \\ \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{c}_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{d}_i}{\hbar}\right)^{\omega_i}\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\delta_i}{\hbar}\right)^{\omega_i}\right)} \right]} \end{pmatrix}, \begin{pmatrix} s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)^{\omega_i}}} e^{i s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{ϡ_1^2}{\hbar^2}\right)^{\omega_i}}}} \\ s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{v}_i}{\hbar}\right)^{\omega_i}\right)} e^{i s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{n}_i}{\hbar}\right)^{\omega_i}\right)}} \end{pmatrix} \right)$$

$$LCCuPFWAAO(Ⱡ, Ⱡ, Ⱡ, \dots, Ⱡ)$$

$$= \left(\begin{pmatrix} \left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{a}^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{b}^2}{\hbar^2}\right)^{\omega_i}}} \right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\alpha^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\beta^2}{\hbar^2}\right)^{\omega_i}}} \right]}, \\ \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{c}}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{d}}{\hbar}\right)^{\omega_i}\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\gamma}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\delta}{\hbar}\right)^{\omega_i}\right)} \right]} \end{pmatrix}, \begin{pmatrix} s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{u}^2}{\hbar^2}\right)^{\omega_i}}} e^{i s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{ϡ^2}{\hbar^2}\right)^{\omega_i}}}} \\ s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{v}}{\hbar}\right)^{\omega_i}\right)} e^{i s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{n}}{\hbar}\right)^{\omega_i}\right)}} \end{pmatrix} \right)$$

$$LCCuPFWAAO(Ł, Ł, Ł, \dots, Ł) = \left(\begin{array}{c}\left(\left[s_{\hbar\sqrt{1-\left(1-\frac{a^2}{\hbar^2}\right)^{\sum_{i=1}^{n}\omega_i}}}, s_{\hbar\sqrt{1-\left(1-\frac{b^2}{\hbar^2}\right)^{\sum_{i=1}^{n}\omega_i}}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\alpha^2}{\hbar^2}\right)^{\sum_{i=1}^{n}\omega_i}}}, s_{\hbar\sqrt{1-\left(1-\frac{\beta^2}{\hbar^2}\right)^{\sum_{i=1}^{n}\omega_i}}}\right]}, \left[s_{\hbar\left(\left(\frac{c}{\hbar}\right)^{\sum_{i=1}^{n}\omega_i}\right)}, s_{\hbar\left(\left(\frac{d}{\hbar}\right)^{\sum_{i=1}^{n}\omega_i}\right)}\right] e^{i\left[s_{\hbar\left(\left(\frac{\gamma}{\hbar}\right)^{\sum_{i=1}^{n}\omega_i}\right)}, s_{\hbar\left(\left(\frac{\delta}{\hbar}\right)^{\sum_{i=1}^{n}\omega_i}\right)}\right]}\right), \\ \left(s_{\hbar\sqrt{1-\left(1-\frac{u^2}{\hbar^2}\right)^{\sum_{i=1}^{n}\omega_i}}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{P}^2}{\hbar^2}\right)^{\sum_{i=1}^{n}\omega_i}}}}, s_{\hbar\left(\left(\frac{v}{\hbar}\right)^{\sum_{i=1}^{n}\omega_i}\right)} e^{is_{\hbar\left(\left(\frac{n}{\hbar}\right)^{\sum_{i=1}^{n}\omega_i}\right)}}\right)\end{array}\right)$$

$$LCCuPFWAAO(Ł, Ł, Ł, \dots, Ł) = \left(\begin{array}{c}\left(\left[s_{\hbar\sqrt{1-\left(1-\frac{a^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{b^2}{\hbar^2}\right)}}\right] e^{i\left[s_{\hbar\sqrt{1-\left(1-\frac{\alpha^2}{\hbar^2}\right)}}, s_{\hbar\sqrt{1-\left(1-\frac{\beta^2}{\hbar^2}\right)}}\right]}, \left[s_{\hbar\left(\left(\frac{c}{\hbar}\right)\right)}, s_{\hbar\left(\left(\frac{d}{\hbar}\right)\right)}\right] e^{i\left[s_{\hbar\left(\left(\frac{\gamma}{\hbar}\right)\right)}, s_{\hbar\left(\left(\frac{\delta}{\hbar}\right)\right)}\right]}\right), \\ \left(s_{\hbar\sqrt{1-\left(1-\frac{u^2}{\hbar^2}\right)}} e^{is_{\hbar\sqrt{1-\left(1-\frac{\mathfrak{P}^2}{\hbar^2}\right)}}}, s_{\hbar\left(\left(\frac{v}{\hbar}\right)\right)} e^{is_{\hbar\left(\left(\frac{n}{\hbar}\right)\right)}}\right)\end{array}\right)$$

$$= \left(\begin{array}{c}\left([s_a, s_b] e^{i[s_\alpha, s_\beta]}, [s_c, s_d] e^{i[s_\gamma, s_\delta]}\right), \\ \left(s_u e^{is_{\mathfrak{P}}}, s_v e^{is_n}\right)\end{array}\right) = Ł.$$

ii). Similarly, we can prove (ii).

Theorem 7: Suppose $Ł_i = \left(\begin{array}{c}\left([s_{a_i}, s_{b_i}] e^{i[s_{\alpha_i}, s_{\beta_i}]}, [s_{c_i}, s_{d_i}] e^{i[s_{\gamma_i}, s_{\delta_i}]}\right), \\ \left(s_{u_i} e^{is_{\mathfrak{P}_i}}, s_{v_i} e^{is_{n_i}}\right)\end{array}\right)$ and $Ł'_i = \left(\begin{array}{c}\left([s_{a'_i}, s_{b'_i}] e^{i[s_{\alpha'_i}, s_{\beta'_i}]}, [s_{c'_i}, s_{d'_i}] e^{i[s_{\gamma'_i}, s_{\delta'_i}]}\right), \\ \left(s_{u'_i} e^{is_{\mathfrak{P}'_i}}, s_{v'_i} e^{is_{n'_i}}\right)\end{array}\right)$ represents two collections of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ and $\omega_i > 0$ such that $a_i \le a'_i, b_i \le b'_i, \alpha_i \le \alpha'_i, \beta_i \le \beta'_i, u_i \le u'_i, \mathfrak{P}_i \le \mathfrak{P}'_i, c'_i \le c_i, d'_i \le d_i, \gamma'_i \le \gamma_i, \delta'_i \le \delta_i, v'_i \le v_i$, and $n'_i \le n_i$ for all i, then

i) $LCCuPFAAO(Ł_1, Ł_2, Ł_3, \dots, Ł_n) \le LCCuPFAAO(Ł'_1, Ł'_2, Ł'_3, \dots, Ł'_n)$.
ii) $LCCuPFGAO(Ł_1, Ł_2, Ł_3, \dots, Ł_n) \le LCCuPFGAO(Ł'_1, Ł'_2, Ł'_3, \dots, Ł'_n)$.
iii) $LCCuPFWAAO(Ł_1, Ł_2, Ł_3, \dots, Ł_n) \le LCCuPFWAAO(Ł'_1, Ł'_2, Ł'_3, \dots, Ł'_n)$.
iv) $LCCuPFWGAO(Ł_1, Ł_2, Ł_3, \dots, Ł_n) \le LCCuPFWGAO(Ł'_1, Ł'_2, Ł'_3, \dots, Ł'_n)$.

Proof: Suppose $\text{Ⱡ}_i = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_i}, \mathfrak{s}_{\mathbb{b}_i}\right]e^{i\left[\mathfrak{s}_{\alpha_i}, \mathfrak{s}_{\beta_i}\right]}, \left[\mathfrak{s}_{\mathbb{c}_i}, \mathfrak{s}_{\mathbb{d}_i}\right]e^{i\left[\mathfrak{s}_{\gamma_i}, \mathfrak{s}_{\delta_i}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_i}e^{i\mathfrak{s}_{\mathfrak{D}_i}}, \mathfrak{s}_{\mathfrak{v}_i}e^{i\mathfrak{s}_{\mathfrak{n}_i}}\right) \end{pmatrix}$ and $\text{Ⱡ}'_i = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}'_i}, \mathfrak{s}_{\mathbb{b}'_i}\right]e^{i\left[\mathfrak{s}_{\alpha'_i}, \mathfrak{s}_{\beta'_i}\right]}, \left[\mathfrak{s}_{\mathbb{c}'_i}, \mathfrak{s}_{\mathbb{d}'_i}\right]e^{i\left[\mathfrak{s}_{\gamma'_i}, \mathfrak{s}_{\delta'_i}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}'_i}e^{i\mathfrak{s}_{\mathfrak{D}'_i}}, \mathfrak{s}_{\mathfrak{v}'_i}e^{i\mathfrak{s}_{\mathfrak{n}'_i}}\right) \end{pmatrix}$ represents two collections of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ such that $\mathbb{a}_i \le \mathbb{a}'_i, \mathbb{b}_i \le \mathbb{b}'_i, \alpha_i \le \alpha'_i, \beta_i \le \beta'_i, \mathfrak{u}_i \le \mathfrak{u}'_i, \mathfrak{D}_i \le \mathfrak{D}'_i, \mathbb{c}'_i \le \mathbb{c}_i, \mathbb{d}'_i \le \mathbb{d}_i, \gamma'_i \le \gamma_i, \delta'_i \le \delta_i, \mathfrak{v}'_i \le \mathfrak{v}_i$, and $\mathfrak{n}'_i \le \mathfrak{n}_i$ for all i, then $\left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right) \le \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right)$ this implies that $\prod_{i=1}^n \left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right) \le \prod_{i=1}^n \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right)$ and $1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right) \le 1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right)$. Hence, we get $\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right)} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right)}$.

Similarly, we may follow the same procedure to get the following inequalities

$$\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{b}_i^2}{\hbar^2}\right)} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{b}'^2_i}{\hbar^2}\right)},$$

$$\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\alpha_i^2}{\hbar^2}\right)} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\alpha'^2_i}{\hbar^2}\right)},$$

$$\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\beta_i^2}{\hbar^2}\right)} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\beta'^2_i}{\hbar^2}\right)},$$

$$\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathfrak{u}_i^2}{\hbar^2}\right)} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathfrak{u}'^2_i}{\hbar^2}\right)},$$

$$\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathfrak{D}_i^2}{\hbar^2}\right)} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathfrak{D}'^2_i}{\hbar^2}\right)}.$$

On the other hand, $\mathbb{c}'_i \le \mathbb{c}_i, \mathbb{d}'_i \le \mathbb{d}_i, \gamma'_i \le \gamma_i, \delta'_i \le \delta_i, \mathfrak{v}'_i \le \mathfrak{v}_i$, and $\mathfrak{n}'_i \le \mathfrak{n}_i$ for all i, then $\frac{\mathbb{c}'_i}{\hbar} \le \frac{\mathbb{c}_i}{\hbar}$ this implies that $\prod_{i=1}^n \frac{\mathbb{c}'_i}{\hbar} \le \prod_{i=1}^n \frac{\mathbb{c}_i}{\hbar}$, and hence we get $\hbar\left(\prod_{i=1}^n \frac{\mathbb{c}'_i}{\hbar}\right) \le \hbar\left(\prod_{i=1}^n \frac{\mathbb{c}_i}{\hbar}\right)$.

Similarly, we can obtain the inequalities $\hbar\left(\prod_{i=1}^n \frac{\mathbb{d}'_i}{\hbar}\right) \le \hbar\left(\prod_{i=1}^n \frac{\mathbb{d}_i}{\hbar}\right)$, $\hbar\left(\prod_{i=1}^n \frac{\gamma'_i}{\hbar}\right) \le \hbar\left(\prod_{i=1}^n \frac{\gamma_i}{\hbar}\right)$, $\hbar\left(\prod_{i=1}^n \frac{\delta'_i}{\hbar}\right) \le \hbar\left(\prod_{i=1}^n \frac{\delta_i}{\hbar}\right)$, $\hbar\left(\prod_{i=1}^n \frac{\mathfrak{v}'_i}{\hbar}\right) \le \hbar\left(\prod_{i=1}^n \frac{\mathfrak{v}_i}{\hbar}\right)$, and $\hbar\left(\prod_{i=1}^n \frac{\mathfrak{n}'_i}{\hbar}\right) \le \hbar\left(\prod_{i=1}^n \frac{\mathfrak{n}_i}{\hbar}\right)$.

Thus, the above inequalities implying that

$$LCCuPFAAO(\text{Ⱡ}_1, \text{Ⱡ}_2, \text{Ⱡ}_3, \dots, \text{Ⱡ}_n) \le LCCuPFAAO(\text{Ⱡ}'_1, \text{Ⱡ}'_2, \text{Ⱡ}'_3, \dots, \text{Ⱡ}'_n).$$

ii) We may prove (ii) using the same procedure.

iii) Suppose $\text{Ⱡ}_i = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}_i}, \mathfrak{s}_{\mathbb{b}_i}\right]e^{i\left[\mathfrak{s}_{\alpha_i}, \mathfrak{s}_{\beta_i}\right]}, \left[\mathfrak{s}_{\mathbb{c}_i}, \mathfrak{s}_{\mathbb{d}_i}\right]e^{i\left[\mathfrak{s}_{\gamma_i}, \mathfrak{s}_{\delta_i}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}_i}e^{i\mathfrak{s}_{\mathfrak{D}_i}}, \mathfrak{s}_{\mathfrak{v}_i}e^{i\mathfrak{s}_{\mathfrak{n}_i}}\right) \end{pmatrix}$ and $\text{Ⱡ}'_i = \begin{pmatrix} \left(\left[\mathfrak{s}_{\mathbb{a}'_i}, \mathfrak{s}_{\mathbb{b}'_i}\right]e^{i\left[\mathfrak{s}_{\alpha'_i}, \mathfrak{s}_{\beta'_i}\right]}, \left[\mathfrak{s}_{\mathbb{c}'_i}, \mathfrak{s}_{\mathbb{d}'_i}\right]e^{i\left[\mathfrak{s}_{\gamma'_i}, \mathfrak{s}_{\delta'_i}\right]}\right), \\ \left(\mathfrak{s}_{\mathfrak{u}'_i}e^{i\mathfrak{s}_{\mathfrak{D}'_i}}, \mathfrak{s}_{\mathfrak{v}'_i}e^{i\mathfrak{s}_{\mathfrak{n}'_i}}\right) \end{pmatrix}$ represents two collections of n LCCuPFNs defined over $\mathbb{S}_{[0,\hbar]}$ such that $\mathbb{a}_i \le \mathbb{a}'_i, \mathbb{b}_i \le \mathbb{b}'_i, \alpha_i \le \alpha'_i, \beta_i \le \beta'_i, \mathfrak{u}_i \le \mathfrak{u}'_i, \mathfrak{D}_i \le \mathfrak{D}'_i, \mathbb{c}'_i \le \mathbb{c}_i, \mathbb{d}'_i \le \mathbb{d}_i, \gamma'_i \le \gamma_i, \delta'_i \le \delta_i, \mathfrak{v}'_i \le \mathfrak{v}_i$, and $\mathfrak{n}'_i \le \mathfrak{n}_i$ for all i, then $\left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right)^{\omega_i} \le \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right)^{\omega_i}$ this implies that $\prod_{i=1}^n \left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right)^{\omega_i} \le \prod_{i=1}^n \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right)^{\omega_i}$ and $1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right)^{\omega_i} \le 1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right)^{\omega_i}$. Hence, we get $\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}_i^2}{\hbar^2}\right)^{\omega_i}} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{a}'^2_i}{\hbar^2}\right)^{\omega_i}}$.

Similarly, we may follow the same procedure to get the following inequalities

$$\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{b}_i^2}{\hbar^2}\right)^{\omega_i}} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\mathbb{b}'^2_i}{\hbar^2}\right)^{\omega_i}},$$

$$\hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\alpha_i^2}{\hbar^2}\right)^{\omega_i}} \le \hbar\sqrt{1 - \prod_{i=1}^n \left(1 - \frac{\alpha'^2_i}{\hbar^2}\right)^{\omega_i}},$$

$$\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\beta_i^2}{\hbar^2}\right)^{\omega_i}} \le \hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{{\beta'}_i^2}{\hbar^2}\right)^{\omega_i}},$$

$$\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)^{\omega_i}} \le \hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{{\mathfrak{u}'}_i^2}{\hbar^2}\right)^{\omega_i}},$$

$$\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{D}_i^2}{\hbar^2}\right)^{\omega_i}} \le \hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{{\mathfrak{D}'}_i^2}{\hbar^2}\right)^{\omega_i}}.$$

On the other hand, ${\mathbb{c}'}_i \le \mathbb{c}_i, {\mathbb{d}'}_i \le \mathbb{d}_i,\ {\gamma'}_i \le \gamma_i,\ {\delta'}_i \le \delta_i, {\mathfrak{v}'}_i \le \mathfrak{v}_i$, and ${\mathfrak{n}'}_i \le \mathfrak{n}_i$ for all i, then $\frac{{\mathbb{c}'}_i}{\hbar} \le \frac{\mathbb{c}_i}{\hbar}$ this implies that $\prod_{i=1}^{n}\frac{{\mathbb{c}'}_i}{\hbar} \le \prod_{i=1}^{n}\frac{\mathbb{c}_i}{\hbar}$, and hence we get $\hbar\left(\prod_{i=1}^{n}\left(\frac{{\mathbb{c}'}_i}{\hbar}\right)^{\omega_i}\right) \le \hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{c}_i}{\hbar}\right)^{\omega_i}\right)$.

Similarly, we can obtain the inequalities $\hbar\left(\prod_{i=1}^{n}\left(\frac{{\mathbb{d}'}_i}{\hbar}\right)^{\omega_i}\right) \le \hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{d}_i}{\hbar}\right)^{\omega_i}\right)$, $\hbar\left(\prod_{i=1}^{n}\left(\frac{{\gamma'}_i}{\hbar}\right)^{\omega_i}\right) \le \hbar\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{\hbar}\right)^{\omega_i}\right)$, $\hbar\left(\prod_{i=1}^{n}\left(\frac{{\delta'}_i}{\hbar}\right)^{\omega_i}\right) \le \hbar\left(\prod_{i=1}^{n}\left(\frac{\delta_i}{\hbar}\right)^{\omega_i}\right)$, $\hbar\left(\prod_{i=1}^{n}\left(\frac{{\mathfrak{v}'}_i}{\hbar}\right)^{\omega_i}\right) \le \hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{v}_i}{\hbar}\right)^{\omega_i}\right)$, and $\hbar\left(\prod_{i=1}^{n}\left(\frac{{\mathfrak{n}'}_i}{\hbar}\right)^{\omega_i}\right) \le \hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{n}_i}{\hbar}\right)^{\omega_i}\right)$.

Thus, the above inequalities implying that

$$LCCuPFWAAO(Ⱡ_1, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n) \le LCCuPFWAAO({Ⱡ'}_1, {Ⱡ'}_2, {Ⱡ'}_3, \dots, {Ⱡ'}_n).$$

iv) We may prove (iv) using the same procedure.

Theorem 8: Let $Ⱡ^- = \left(\begin{matrix}\left([s_{\mathbb{a}^-}, s_{\mathbb{b}^-}]e^{i[s_{\alpha^-}, s_{\beta^-}]}, [s_{\mathbb{c}^-}, s_{\mathbb{d}^-}]e^{i[s_{\gamma^-}, s_{\delta^-}]}\right),\\ (s_{\mathfrak{u}^-}\text{-}e^{is_{\mathfrak{D}^-}}, s_{\mathfrak{v}^-}\text{-}e^{is_{\mathfrak{n}^-}})\end{matrix}\right)$ and $Ⱡ^+ = \left(\begin{matrix}\left([s_{\mathbb{a}^+}, s_{\mathbb{b}^+}]e^{i[s_{\alpha^+}, s_{\beta^+}]}, [s_{\mathbb{c}^+}, s_{\mathbb{d}^+}]e^{i[s_{\gamma^+}, s_{\delta^+}]}\right),\\ (s_{\mathfrak{u}^+}e^{is_{\mathfrak{D}^+}}, s_{\mathfrak{v}^+}e^{is_{\mathfrak{n}^+}})\end{matrix}\right)$ where $\mathbb{a}^- = min_i\{\mathbb{a}_i\}, \mathbb{b}^- = min_i\{\mathbb{b}_i\}, \alpha^- = min_i\{\alpha_i\}, \beta^- = min_i\{\beta_i\}, \mathfrak{u}^- = min_i\{\mathfrak{u}_i\}, \mathfrak{D}^- = min_i\{\mathfrak{D}_i\}, \mathbb{c}^- = max_i\{\mathbb{c}_i\}, \mathbb{d}^- = max_i\{\mathbb{d}_i\}, \gamma^- = max_i\{\gamma_i\}, \delta^- = max_i\{\delta_i\}, \mathfrak{v}^- = max_i\{\mathfrak{v}_i\}, \mathfrak{n}^- = max_i\{\mathfrak{n}_i\}$ and $\mathbb{a}^+ = max_i\{\mathbb{a}_i\}, \mathbb{b}^+ = max_i\{\mathbb{b}_i\}, \alpha^+ = max_i\{\alpha_i\}, \beta^+ = max_i\{\beta_i\}, \mathfrak{u}^+ = max_i\{\mathfrak{u}_i\}, \mathfrak{D}^+ = max_i\{\mathfrak{D}_i\}, \mathbb{c}^+ = min_i\{\mathbb{c}_i\}, \mathbb{d}^+ = min_i\{\mathbb{d}_i\}, \gamma^+ = min_i\{\gamma_i\}, \delta^+ = min_i\{\delta_i\}, \mathfrak{v}^+ = min_i\{\mathfrak{v}_i\}, \mathfrak{n}^+ = min_i\{\mathfrak{n}_i\}$, then

i) $Ⱡ^- \le LCCuPFAAO(Ⱡ_1, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n) \le Ⱡ^+$.
ii) $Ⱡ^- \le LCCuPFAAO(Ⱡ_1, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n) \le Ⱡ^+$.
iii) $Ⱡ^- \le LCCuPFAAO(Ⱡ_1, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n) \le Ⱡ^+$.
iv) $Ⱡ^- \le LCCuPFAAO(Ⱡ_1, Ⱡ_2, Ⱡ_3, \dots, Ⱡ_n) \le Ⱡ^+$.

Proof: The Proof of Theorem 8 follows by Theorem 7.

5. Decision-Making Approach under Linguistic Complex Cubic Pythagorean Fuzzy Sets

Decision-making problems play a crucial role in real-life applications, as individuals, businesses, and governments frequently encounter situations where they must choose between a set of alternatives with specific constraints. For example, in the context of financial investments, individuals are required to decide about the ratio of available funds to be allocated among debt, equity or hybrid securities based on their relative pros and cons. Adopting a new technology, entering into a new geographic area or launching a new product are some of the decision-making problems that are faced by businesses and these factors require a considerable amount of data for careful evaluation of profitability, competition, and long-term sustainability. Physicians – in hospitals, finalize crucial decisions about treatment of patients by balancing risks and recovery time to ensure patient's well-being. Students as well as professionals face education and career related decisions, for instance selecting courses, HEIs, or career paths, based on interests, job opportunities, lifelong learning and growth prospects. Governments have to tackle huge problems while formulating and subsequently implementing policies, allocating resources, or addressing crises. It requires maintaining reasonable trade-offs between economic growth, public welfare, and environmental sustainability. Even in daily life, simple tasks like choosing a route to work, procurement of grocery items or managing day to day expenses involve decision-making under constraints. In all these cases, structured decision-making procedures help evaluate alternatives systematically, manage uncertainty, and ensure that choices are rational, effective, and aligned with long-term goals.

Here, we establish a DM approach within the framework of LCCuPFSs. We employ the newly defined approaches such as LCCuPFAAO, LCCuPFGAO, LCCuPFWAAO, LCCuPFWGAO, and S.F or A.F to develop the proposed algorithm. The flowchart of the DM technique based on the proposed approaches is demonstrated below (Figure 1):

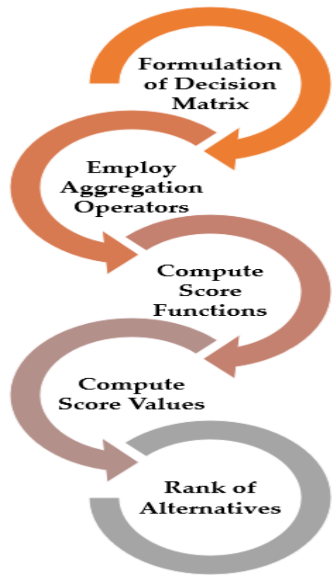

Figure 1: Flowchart of the DM Approach.

The step-by-step procedure for DM is discussed as in the following.

5.1 Decision-Making Algorithm

Suppose $\zeta = \{\zeta_1, \zeta_2, \zeta_3, \dots, \zeta_n\}$ identifies a family of n distinct alternatives. The decision is made based on these defined alternatives. Assume that $\mathfrak{h} = \{\mathfrak{h}_1, \mathfrak{h}_2, \mathfrak{h}_3, \dots, \mathfrak{h}_m\}$ denotes a collection of m attributes or criteria used to assess the given alternatives. A set of expert's team is represented by Ч which contributes a critical role in the evaluation of both quantitative and qualitative criteria. Let $\omega = \{\omega_1, \omega_2, \omega_3, \dots, \omega_m\}$ be a set of positive weight vectors which indicate the relative importance of criteria. To evaluate the alternatives $\zeta_i; i = 1,2,3, \dots, n$, based on the attributes $\mathfrak{h}_j; j = 1,2,3, \dots, m$, within the framework of LCCuPFSs, the proposed steps are given below:

Step 1: Step 1 provides the formulation of decision matrix (DM) which plays a crucial role in summarizing the evaluation information provided by various experts. The DM helps to arrange the assessment of all alternatives based on the defined attributes in a tabular and comparable form. Suppose $\zeta_i; i = 1,2,3, \dots, n$, be possible alternative and $\mathfrak{h}_j; j = 1,2,3, \dots, m$, denotes their defined criteria then the DM within the framework of LCCuPFSs can be defined mathematically as:

$$[D^{\text{Ч}_k}]_{n\times m} = \left[\left(\begin{matrix}\left(\left[\mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{a}_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{b}_{ij}}\right] e^{i\left[\mathfrak{s}^{\text{Ч}_k}{}_{\alpha_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\beta_{ij}}\right]}, \left[\mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{c}_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{d}_{ij}}\right] e^{i\left[\mathfrak{s}^{\text{Ч}_k}{}_{\gamma_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\delta_{ij}}\right]}\right), \\ \left(\mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{u}_{ij}} e^{i\mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{D}_{ij}}}, \mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{v}_{ij}} e^{i\mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{n}_{ij}}}\right)\end{matrix}\right)\right]_{n\times m},$$

where $\left(\begin{matrix}\left(\left[\mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{a}_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{b}_{ij}}\right] e^{i\left[\mathfrak{s}^{\text{Ч}_k}{}_{\alpha_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\beta_{ij}}\right]}, \left[\mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{c}_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\mathbb{d}_{ij}}\right] e^{i\left[\mathfrak{s}^{\text{Ч}_k}{}_{\gamma_{ij}}, \mathfrak{s}^{\text{Ч}_k}{}_{\delta_{ij}}\right]}\right), \\ \left(\mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{u}_{ij}} e^{i\mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{D}_{ij}}}, \mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{v}_{ij}} e^{i\mathfrak{s}^{\text{Ч}_k}{}_{\mathfrak{n}_{ij}}}\right)\end{matrix}\right)$ denotes the assessment of alternative ζ_i under criterion $\mathfrak{h}_j$ provided by Ч_k expert.

Step 2: In step 2, we employ the proposed AOs to aggregate the information provided by various experts. The aggregation process is critical since different experts may give the information based on their experience and expertise. The proposed AOs are employed to combine these assessments into a single LCCuPFN framework. The utilization of proposed AOs depends on the nature of DM problem. If the experts ignore the relative importance of the criterion, then the LCCuPFAAO or LCCuPFGAO are employed which are given below:

$$LCCuPFAAO(Ɫ_i, Ɫ_2, Ɫ_3, \dots, Ɫ_n) = \left(\left(\begin{array}{c} \left[s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{a}_i^2}{\hbar^2}\right)}}, s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{b}_i^2}{\hbar^2}\right)}} \right] e^{i\left[s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{\hbar^2}\right)}}, s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\beta_i^2}{\hbar^2}\right)}} \right]}, \\ \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{c}_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{d}_i}{\hbar}\right)\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\delta_i}{\hbar}\right)\right)} \right]} \end{array} \right), \left(\begin{array}{c} s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)}} e^{i s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\text{Ђ}_1^2}{\hbar^2}\right)}}} \\ s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{v}_i}{\hbar}\right)\right)} e^{i s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{n}_i}{\hbar}\right)\right)}} \end{array} \right) \right),$$

or

$$LCCuPFAAO(Ɫ_i, Ɫ_2, Ɫ_3, \dots, Ɫ_n) = \left(\left(\begin{array}{c} \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{a}_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{b}_i}{\hbar}\right)\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\alpha_i}{\hbar}\right)\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\beta_i}{\hbar}\right)\right)} \right]}, \\ \left[s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{c}_i^2}{\hbar^2}\right)}}, s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{d}_i^2}{\hbar^2}\right)}} \right] e^{i\left[s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\gamma_i^2}{\hbar^2}\right)}}, s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\delta_i^2}{\hbar^2}\right)}} \right]} \end{array} \right), \left(\begin{array}{c} s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{u}_i}{\hbar}\right)\right)} e^{i s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\text{Ђ}_i}{\hbar}\right)\right)}} \\ s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{v}_i^2}{\hbar^2}\right)}} e^{i s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{n}_i^2}{\hbar^2}\right)}}} \end{array} \right) \right).$$

If the experts consider the relative importance of criterion, then the LCCuPFWAAO or LCCuPFWGAO are employed to aggregate the data. They are given below:

$$LCCuPFWAAO(Ɫ_i, Ɫ_2, Ɫ_3, \dots, Ɫ_n)$$

$$= \left(\left(\begin{array}{c} \left[s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{a}_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{b}_i^2}{\hbar^2}\right)^{\omega_i}}} \right] e^{i\left[s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\alpha_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\beta_i^2}{\hbar^2}\right)^{\omega_i}}} \right]}, \\ \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{c}_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{d}_i}{\hbar}\right)^{\omega_i}\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\gamma_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\delta_i}{\hbar}\right)^{\omega_i}\right)} \right]} \end{array} \right), \left(\begin{array}{c} s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{u}_i^2}{\hbar^2}\right)^{\omega_i}}} e^{i s_{\sqrt[\hbar]{1-\prod_{i=1}^{n}\left(1-\frac{\text{Ђ}_1^2}{\hbar^2}\right)^{\omega_i}}}} \\ s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{v}_i}{\hbar}\right)^{\omega_i}\right)} e^{i s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{n}_i}{\hbar}\right)^{\omega_i}\right)}} \end{array} \right) \right)$$

or

$$LCCuPFWGAO(\text{Ⱡ}_i, \text{Ⱡ}_2, \text{Ⱡ}_3, \ldots, \text{Ⱡ}_n)$$

$$= \left(\begin{array}{c} \left(\begin{array}{c} \left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{a}_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathbb{b}_i}{\hbar}\right)^{\omega_i}\right)} \right] e^{i\left[s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\alpha_i}{\hbar}\right)^{\omega_i}\right)}, s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\beta_i}{\hbar}\right)^{\omega_i}\right)} \right]}, \\ \left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{c}_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathbb{d}_i^2}{\hbar^2}\right)^{\omega_i}}} \right] e^{i\left[s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\gamma_i^2}{\hbar^2}\right)^{\omega_i}}}, s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\delta_i^2}{\hbar^2}\right)^{\omega_i}}} \right]} \end{array} \right), \\ \left(\begin{array}{c} s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{u}_i}{\hbar}\right)^{\omega_i}\right)} e^{i s_{\hbar\left(\prod_{i=1}^{n}\left(\frac{\mathfrak{H}_i}{\hbar}\right)^{\omega_i}\right)}} \\ s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{v}_i^2}{\hbar^2}\right)^{\omega_i}}} e^{i s_{\hbar\sqrt{1-\prod_{i=1}^{n}\left(1-\frac{\mathfrak{n}_1^2}{\hbar^2}\right)^{\omega_i}}}} \end{array} \right) \end{array} \right).$$

Step 3: Step 3 computes the S.Fs of the aggregated information to get a single comparable value for each alternative. After aggregating the LCCuPF information provided by different experts, the S.F is employed to obtain a single numerical value. The S.F is demonstrated as:

$$S.F(\text{ʒ}) = s_{\sqrt{\frac{6\hbar^2+\mathbb{a}^2-\mathbb{c}^2+\mathbb{b}^2-\mathbb{d}^2+\alpha^2-\gamma^2+\beta^2-\delta^2+\mathfrak{u}^2-\mathfrak{v}^2+\mathfrak{H}^2-\mathfrak{n}^2}{12}}}.$$

If the S.Fs of two alternatives has same values then the A.F is employed to rank the alternatives. The A.F is given as:

$$A.F(\text{ʒ}) = s_{\sqrt{\frac{\mathbb{a}^2+\mathbb{c}^2+\mathbb{b}^2+\mathbb{d}^2+\alpha^2+\gamma^2+\beta^2+\delta^2+\mathfrak{u}^2+\mathfrak{v}^2+\mathfrak{H}^2+\mathfrak{n}^2}{6}}}.$$

Step 4: When multiple experts take part in DM process, each expert contributes an assessment for every alternative and get one score function per expert. In case of k −experts, we obtain

$$S.F^{(\text{Ч}_1)}(\text{ʒ}_i), S.F^{(\text{Ч}_2)}(\text{ʒ}_i), \ldots., S.F^{(\text{Ч}_k)}(\text{ʒ}_i),$$

where $S.F^{(\text{Ч}_k)}(\text{ʒ}_i)$ represents a S.F for alternative ʒ_i computed from expert Ч_k. To obtain a single comparable numerical value $S.F(\text{Ⱡ}_i)$ for each alternative Ⱡ_i, we employ the following formula:

$$S.V(\text{ʒ}_i) = \frac{\sum_{k=1}^{n} S.F^{(\text{Ч}_k)}(\text{ʒ}_i)}{n\hbar}.$$

Step 5: In step 5, we rank the alternatives based on computed score value $S.V(\text{ʒ}_i); i = 1,2, \ldots, n.$ An alternative Ⱡ_i with higher score value identifies a better alternative, and an alternative Ⱡ_i with lower score value indicates a worst alternative.

6. Case Study

In this section, we discuss stock market DM problems using the proposed DM procedure.

Investing in the stock market is a decision-making process which goes far beyond the simple act of buying and selling shares and this is the reason it is termed as multi-dimensional DM process. The decisions pertaining to investment in stock exchanges require strategic thinking due to the fact that share prices goes up and down because of various political, economic and even psychological factors and all these factors usually intensify uncertainty surrounding investment decisions. Investors are required to vigilantly evaluate alternatives (various types of shares, sectors and strategies) against multiple criteria (e.g., dividend, capital gain, risk, liquidity and returns in foreseeable future). The main issue is that returns in the future are never ever guaranteed, therefore making DM process significantly blur, therefore it compels investors to compare risk-return trade-offs and then finalize a decision on the basis of best possible alternative.

6.1 Alternatives in Stock Market Decisions

Stock market presents different alternatives to investors depending upon resources available with them, goals they want to achieve and risk tolerance level. Some of the alternatives available in the stock exchanges are given below:

1. Purchasing stocks of various firms "$ȥ_1$": Investors may choose shares of a company that either have stable performance, strong brand loyalty and offer consistent income in the form of dividends or may choose a firm which offers innovative products but have volatile returns.
2. Buy and hold strategy "$ȥ_2$": Instead of buying or selling shares, investors may decide to purchase and then hold shares in expectation of future price appreciation. For instance, an investor might opt to hold shares of a firm which is currently facing a terrible run but experts are in the view that upcoming innovative product launch will help the firm to recover and this investment will offer better yields and / or capital gains.
3. Investing in mutual funds or ETFs "$ȥ_3$": Instead of buying shares of a specific company, investors may utilize their surplus money to invest in either mutual funds or ETFs. Investment in this type provides diversification to the investors and at the same time reduces risk.
4. Sector – wise investment "$ȥ_4$": Investors have another alternative i.e., to invest in a specific sector / industry (e.g., pharmaceutical, technology, fertilizer, renewable energy, finance, steel or fashion industry etc.). This type of investment helps the investor (s) to gain sizeable profit due to the fact that a sector (at a given point in time) may provide yields higher than expected primarily due to macro level factors.

6.2 Criteria for Stock Market Decisions

An optimal alternative when selected depends upon "criteria" which in turn affects outcomes of an investment.

1. Anticipated Return (Profitability) "$ɧ_1$": It refers to the potential profit an investor expects from a stock depending upon its most recent earnings, past performance, and market trends. In order to finalize a financial decision, this criterion serves as a benchmark against which one can gauge future yield as well as capital gain. For instance, a company which often pays higher dividends or expected to pay high dividends in the future attracts investors that are looking for high profitability.
2. Risk or Volatility "$ɧ_1$": It represents the degree to which a stock's price fluctuates over time, reflecting the level of uncertainty associated with the investment. A highly volatile stock exhibits frequent as well as significant ups and downs in its market price, which offers higher potential returns but at the same time increases the risk of loss. Contrary to this, stocks with low volatility remains stable and predictable over time, making them suitable for risk-averse investors. For example, a well reputed firm which has spent a lot of time in the market is considered less volatile as compared to smaller startups, which makes the former a safer option for conservative investors looking for consistency / stability.
3. Liquidity "$ɧ_1$": Liquidity of an asset / investment has two pre-requisites i.e., i) how quickly and easily a stock can be bought or sold in the market and ii) this process shouldn't create significant difference in sale price and purchase price. Shares with low liquidity may take more time and efforts to sell and may results in huge price difference, whereas highly liquid stocks allow investors to buy or sell securities smoothly, offering flexibility and reduce transaction cost. For example, firms with strong brand loyalty are considered highly liquid and enable investors to trade them with ease and minimal price concession.
4. Time Horizon "$ɧ_1$": It refers to the time span during which investor will retain securities and then finally liquidates them to unlock funds. It plays a crucial role in shaping investment strategies as long-term investors aim for steady growth and stability whereas short-term traders typically prioritize quick gains from volatile stocks. For instance, a retired person may prefer a company on the basis of consistency because it declares dividends regularly while a day trader might invest in highly volatile firm to take advantage of its short-term price swings.
5. Ethical Considerations "$ɧ_1$": It is sometimes referred to as Environmental, Social, and Governance (ESG) factors and highlights the importance of corporate responsibility and sustainability in financial decision-making. In the market, there are investors who prioritize companies that demonstrate ethical practices, environmental care, and strong governance, even if these choices come with slightly lower immediate returns. An investor may choose renewable energy firms over fossil fuel companies thereby focusing on environmental, social and governance issues before finalizing financial decisions, thus such type of decisions ensure sustainability as well long-term grow

Now, we employ the proposed DM algorithm for solving stock market DM problems. We consider the DM problem under these alternatives and their defined criteria.

Suppose $ȥ = \{ȥ_1, ȥ_2, ȥ_3, ȥ_4, ȥ_5\}$ identifies a collection of five distinct alternatives where $ȥ_1, ȥ_2, ȥ_3$, and $ȥ_4$ represent purchasing stocks of various firms, buy and hold strategy, investing in mutual funds or ETFs, and sector – wise investment, respectively. Assume that $ɧ = \{ɧ_1, ɧ_2, ɧ_3, ɧ_4, ɧ_5\}$ denotes a collection of m attributes or criteria used to assess the given alternatives where anticipated return (Profitability), risk or volatility, liquidity, time horizon, and ethical considerations, respectively. Let $Ч = \{Ч_1, Ч_2\}$ be a team of two experts who take part in DM process. Now, we choose a better alternative $ʐ_i; i = 1,2,3,4,5$, using the proposed DM algorithm. The detailed steps to select a best alternative are demonstrated below:

Step 1: In step 1, we arrange the assessment in the form of D-matrices, provided by experts $Ч_1$ and $Ч_2$ for alternatives $ʐ_i; i = 1,2,3,4,5$, based on the defined attributes $ɧ_j; j = 1,2,3,4,5$. Tables 1 and 2 represent the evaluated D-matrices within the framework of LCCuPFSs defined on $\mathfrak{T}_{[0,14]}$.

Table 1: LCCuPF Information Provided by Expert $Ч_1$

Criteria/ Alternatives	$ʐ_1$	$ʐ_2$	$ʐ_3$	$ʐ_4$
$ɧ_1$	$\begin{pmatrix}\begin{pmatrix}[s_2,s_8]e^{i[s_3,s_7]},\\ [s_4,s_9]e^{i[s_2,s_5]}\end{pmatrix}\\ (s_4e^{is_6},s_8e^{is_7})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_{10}]e^{i[s_1,s_6]},\\ [s_4,s_5]e^{i[s_3,s_8]}\end{pmatrix}\\ (s_5e^{is_{11}},s_2e^{is_5})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_7]e^{i[s_5,s_8]},\\ [s_1,s_{10}]e^{i[s_4,s_9]}\end{pmatrix}\\ (s_5e^{is_7},s_2e^{is_6})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_{10}]e^{i[s_4,s_9]},\\ [s_2,s_3]e^{i[s_3,s_6]}\end{pmatrix}\\ (s_2e^{is_9},s_4e^{is_{10}})\end{pmatrix}$
$ɧ_2$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_5]e^{i[s_5,s_{10}]},\\ [s_1,s_3]e^{i[s_4,s_8]}\end{pmatrix}\\ (s_6e^{is_9},s_5e^{is_4})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_{11}]e^{i[s_2,s_{12}]},\\ [s_2,s_4]e^{i[s_1,s_3]}\end{pmatrix}\\ (s_9e^{is_3},s_4e^{is_8})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_7]e^{i[s_5,s_{12}]},\\ [s_2,s_5]e^{i[s_1,s_4]}\end{pmatrix}\\ (s_3e^{is_8},s_5e^{is_9})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_6]e^{i[s_2,s_5]},\\ [s_1,s_6]e^{i[s_3,s_7]}\end{pmatrix}\\ (s_5e^{is_{10}},s_4e^{is_8})\end{pmatrix}$
$ɧ_3$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_6]e^{i[s_2,s_6]},\\ [s_1,s_5]e^{i[s_4,s_7]}\end{pmatrix}\\ (s_6e^{is_2},s_8e^{is_5})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_{10}]e^{i[s_5,s_8]},\\ [s_3,s_8]e^{i[s_7,s_{10}]}\end{pmatrix}\\ (s_5e^{is_8},s_2e^{is_{10}})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_7,s_9]e^{i[s_2,s_5]},\\ [s_1,s_5]e^{i[s_4,s_6]}\end{pmatrix}\\ (s_3e^{is_{12}},s_6e^{is_2})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_5]e^{i[s_4,s_8]},\\ [s_2,s_8]e^{i[s_5,s_{10}]}\end{pmatrix}\\ (s_3e^{is_9},s_2e^{is_6})\end{pmatrix}$
$ɧ_4$	$\begin{pmatrix}\begin{pmatrix}[s_1,s_3]e^{i[s_2,s_5]},\\ [s_5,s_8]e^{i[s_4,s_8]}\end{pmatrix}\\ (s_{10}e^{is_5},s_4e^{is_8})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_9]e^{i[s_2,s_5]},\\ [s_1,s_8]e^{i[s_3,s_6]}\end{pmatrix}\\ (s_6e^{is_8},s_2e^{is_5})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_4]e^{i[s_1,s_6]},\\ [s_5,s_9]e^{i[s_3,s_8]}\end{pmatrix}\\ (s_1e^{is_{10}},s_5e^{is_8})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_7]e^{i[s_2,s_5]},\\ [s_1,s_5]e^{i[s_3,s_6]}\end{pmatrix}\\ (s_{10}e^{is_{12}},s_3e^{is_4})\end{pmatrix}$
$ɧ_5$	$\begin{pmatrix}\begin{pmatrix}[s_6,s_9]e^{i[s_2,s_5]},\\ [s_1,s_3]e^{i[s_5,s_6]}\end{pmatrix}\\ (s_2e^{is_9},s_{12}e^{is_8})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_{12}]e^{i[s_2,s_4]},\\ [s_1,s_2]e^{i[s_4,s_6]}\end{pmatrix}\\ (s_5e^{is_8},s_3e^{is_5})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_6]e^{i[s_1,s_8]},\\ [s_2,s_8]e^{i[s_4,s_6]}\end{pmatrix}\\ (s_{10}e^{is_3},s_5e^{is_8})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_2,s_8]e^{i[s_3,s_7]},\\ [s_4,s_9]e^{i[s_2,s_5]}\end{pmatrix}\\ (s_4e^{is_6},s_8e^{is_7})\end{pmatrix}$

Table 2: LCCuPF Information Provided by Expert $Ч_2$

Criteria/ Alternatives	$ʐ_1$	$ʐ_2$	$ʐ_3$	$ʐ_4$
$ɧ_1$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_6]e^{i[s_1,s_5]},\\ [s_2,s_6]e^{i[s_3,s_8]}\end{pmatrix}\\ (s_1e^{is_8},s_5e^{is_9})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_8]e^{i[s_2,s_8]},\\ [s_5,s_7]e^{i[s_1,s_5]}\end{pmatrix}\\ (s_3e^{is_{12}},s_1e^{is_3})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_{10}]e^{i[s_2,s_7]},\\ [s_3,s_8]e^{i[s_2,s_6]}\end{pmatrix}\\ (s_9e^{is_5},s_5e^{is_8})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_{12}]e^{i[s_2,s_8]},\\ [s_1,s_2]e^{i[s_4,s_5]}\end{pmatrix}\\ (s_4e^{is_6},s_5e^{is_8})\end{pmatrix}$
$ɧ_2$	$\begin{pmatrix}\begin{pmatrix}[s_6,s_8]e^{i[s_3,s_7]},\\ [s_2,s_5]e^{i[s_1,s_8]}\end{pmatrix}\\ (s_4e^{is_{10}},s_9e^{is_3})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_9]e^{i[s_3,s_9]},\\ [s_4,s_6]e^{i[s_2,s_5]}\end{pmatrix}\\ (s_{11}e^{is_5},s_5e^{is_{10}})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_9]e^{i[s_3,s_8]},\\ [s_4,s_7]e^{i[s_3,s_6]}\end{pmatrix}\\ (s_6e^{is_4},s_8e^{is_8})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_2,s_7]e^{i[s_4,s_8]},\\ [s_3,s_9]e^{i[s_1,s_5]}\end{pmatrix}\\ (s_8e^{is_7},s_6e^{is_5})\end{pmatrix}$
$ɧ_3$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_7]e^{i[s_1,s_9]},\\ [s_3,s_7]e^{i[s_2,s_5]}\end{pmatrix}\\ (s_8e^{is_4},s_6e^{is_9})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_6,s_{12}]e^{i[s_2,s_6]},\\ [s_3,s_4]e^{i[s_4,s_8]}\end{pmatrix}\\ (s_7e^{is_{10}},s_4e^{is_6})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_{10}]e^{i[s_5,s_{10}]},\\ [s_3,s_5]e^{i[s_2,s_8]}\end{pmatrix}\\ (s_6e^{is_7},s_8e^{is_5})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_9]e^{i[s_2,s_6]},\\ [s_1,s_7]e^{i[s_2,s_{11}]}\end{pmatrix}\\ (s_6e^{is_7},s_5e^{is_8})\end{pmatrix}$
$ɧ_4$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_5]e^{i[s_4,s_6]},\\ [s_2,s_9]e^{i[s_3,s_9]}\end{pmatrix}\\ (s_9e^{is_8},s_5e^{is_{10}})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_3,s_7]e^{i[s_4,s_8]},\\ [s_3,s_{10}]e^{i[s_1,s_9]}\end{pmatrix}\\ (s_8e^{is_{10}},s_5e^{is_3})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_5,s_8]e^{i[s_3,s_9]},\\ [s_2,s_7]e^{i[s_3,s_6]}\end{pmatrix}\\ (s_3e^{is_8},s_7e^{is_{10}})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_6,s_{10}]e^{i[s_1,s_7]},\\ [s_3,s_4]e^{i[s_2,s_9]}\end{pmatrix}\\ (s_{12}e^{is_{10}},s_3e^{is_3})\end{pmatrix}$
$ɧ_5$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_7]e^{i[s_4,s_8]},\\ [s_3,s_9]e^{i[s_2,s_6]}\end{pmatrix}\\ (s_5e^{is_7},s_{10}e^{is_6})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_1,s_{10}]e^{i[s_4,s_6]},\\ [s_3,s_5]e^{i[s_5,s_{10}]}\end{pmatrix}\\ (s_7e^{is_{10}},s_7e^{is_4})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_1,s_4]e^{i[s_3,s_6]},\\ [s_4,s_7]e^{i[s_2,s_9]}\end{pmatrix}\\ (s_8e^{is_6},s_{10}e^{is_5})\end{pmatrix}$	$\begin{pmatrix}\begin{pmatrix}[s_4,s_6]e^{i[s_5,s_7]},\\ [s_1,s_9]e^{i[s_2,s_8]}\end{pmatrix}\\ (s_6e^{is_7},s_8e^{is_5})\end{pmatrix}$

Table 3 represents positive weight vectors assigned by experts $Ч_1$ and $Ч_2$.

Table 3: Positive Weight Vectors Assigned by Experts $Ч_1$ And $Ч_2$.

Criteria/Alternatives	$Ч_1$	$Ч_2$
$ɧ_1$	$\omega_1 = 0.3$	$\omega_1 = 0.2$
$ɧ_2$	$\omega_2 = 0.1$	$\omega_2 = 0.1$
$ɧ_3$	$\omega_3 = 0.2$	$\omega_3 = 0.3$
$ɧ_4$	$\omega_4 = 0.3$	$\omega_4 = 0.3$
$ɧ_5$	$\omega_5 = 0.1$	$\omega_5 = 0.1$

6.3 Decision-Making Process under LCCuPFAAO or LCCuPFGAO

Step 2: In step 2, we employ LCCuPFAAO to aggregate the information provided in Tables 1 and 2. The proposed AOs are employed to combine these assessments into a single LCCuPFN framework. Tables 4 and 5 denote the aggregated values of the Tables 1 and 2 by LCCuPFAAO.

Table 4: Aggregated Values of the Table 1 by LCCuPFAAO.

Alternatives	Aggregated Values
$ʐ_1$	$\left(\begin{pmatrix} [s_{7.4}, s_{12}]e^{i[s_{6.5}, s_{12.3}]}, \\ [s_{0.001}, s_{0.1}]e^{i[s_{0.02}, s_{0.34}]} \end{pmatrix} \\ (s_{11.8}e^{is_{12.2}}, s_{0.4}e^{is_{0.23}})\right)$
$ʐ_2$	$\left(\begin{pmatrix} [s_{8.7}, s_{13.9}]e^{i[s_{6}, s_{13.2}]}, \\ [s_{0.001}, s_{0.1}]e^{i[s_{0.01}, s_{0.22}]} \end{pmatrix} \\ (s_{11.6}e^{is_{13.2}}, s_{0.002}e^{is_{0.26}})\right)$
$ʐ_3$	$\left(\begin{pmatrix} [s_{8.9}, s_{12.1}]e^{i[s_{7.2}, s_{13.4}]}, \\ [s_{0.001}, s_{0.47}]e^{i[s_{0.005}, s_{0.27}]} \end{pmatrix} \\ (s_{11}e^{is_{13.6}}, s_{0.04}e^{is_{0.18}})\right)$
$ʐ_4$	$\left(\begin{pmatrix} [s_{8.2}, s_{12.7}]e^{i[s_{6.7}, s_{12.3}]}, \\ [s_{0.004}, s_{0.17}]e^{i[s_{0.007}, s_{0.33}]} \end{pmatrix} \\ (s_{11.1}e^{is_{13.7}}, s_{0.02}e^{is_{0.35}})\right)$

Table 5: Aggregated Values of the Table 1 by LCCuPFAAO.

Alternatives	Aggregated Values
$ʐ_1$	$\left(\begin{pmatrix} [s_{9.1}, s_{12}]e^{i[s_{6.3}, s_{12.4}]}, \\ [s_{0.002}, s_{0.15}]e^{i[s_{0.001}, s_{0.45}]} \end{pmatrix} \\ (s_{11.6}e^{is_{12.9}}, s_{0.35}e^{is_{1.01}})\right)$
$ʐ_2$	$\left(\begin{pmatrix} [s_{7.9}, s_{13.7}]e^{i[s_{6.7}, s_{12.7}]}, \\ [s_{0.01}, s_{0.22}]e^{i[s_{0.001}, s_{0.47}]} \end{pmatrix} \\ (s_{13}e^{is_{13.8}}, s_{0.02}e^{is_{0.06}})\right)$
$ʐ_3$	$\left(\begin{pmatrix} [s_{8.8}, s_{13.4}]e^{i[s_{7.1}, s_{13.1}]}, \\ [s_{0.01}, s_{0.36}]e^{i[s_{0.002}, s_{0.4}]} \end{pmatrix} \\ (s_{12.1}e^{is_{11.5}}, s_{0.58}e^{is_{0.42}})\right)$
$ʐ_4$	$\left(\begin{pmatrix} [s_{8.4}, s_{13.7}]e^{i[s_{6.8}, s_{12.5}]}, \\ [s_{0.001}, s_{0.12}]e^{i[s_{0.001}, s_{0.52}]} \end{pmatrix} \\ (s_{13.2}e^{is_{12.8}}, s_{0.09}e^{is_{0.33}})\right)$

Step 3: In step 3, we compute the S.Fs of the aggregated information provided in Tables 4 & 5 to get a single numerical value for each alternative. We employ the following S.F to obtain a single numerical value.

$$S.F(ʐ) = s_{\sqrt{\frac{6\hbar^2 + \mathbb{a}^2 - \mathbb{c}^2 + \mathbb{b}^2 - \mathbb{d}^2 + \alpha^2 - \gamma^2 + \beta^2 - \delta^2 + \mathfrak{u}^2 - \mathfrak{v}^2 + ϸ^2 - \mathfrak{n}^2}{12}}}.$$

After employing the S.F, Tables 6 & 7 represent their numerical score values.

Table 6: Numerical Values of the Table 4 by LCCuPFAAO.

Alternatives	Score Functions
ζ_1	$s_{10.15}$
ζ_2	$s_{10.17}$
ζ_3	$s_{10.17}$
ζ_4	$s_{10.16}$

Table 7: Numerical Values of the Table 5 by LCCuPFAAO.

Alternatives	Score Functions
ζ_1	$s_{10.15}$
ζ_2	$s_{10.18}$
ζ_3	$s_{10.17}$
ζ_4	$s_{10.17}$

Step 4: In step 4, we compute the score values for alternatives $\zeta_i; i = 1,2,3,4$. We employ the following formula to obtain the score values:

$$S.V(\zeta_i) = s_{\frac{\sum_{k=1}^{2} S.F^{(q_k)}(\zeta_i)}{2\hbar}}.$$

Table 8 represents the score values for alternatives $\zeta_i; i = 1,2,3,4$.

Table 8: Score Values by LCCuPFAAO.

Alternatives	Score Values	Rank
ζ_1	$s_{0.724}$	4
ζ_2	$s_{0.727}$	1
ζ_3	$s_{0.726}$	2
ζ_4	$s_{0.725}$	3

Step 5: In step 5, we rank the alternatives based on their score values. Clearly, the alternative ζ_2 has maximum score value. Thus, ζ_2 is the best option among the given alternatives.

6.4 Decision-Making Process under LCCuPFWAAO or LCCuPFWGAO

Now, we employ LCCuPFAAO to aggregate the information provided in Tables 1 and 2. Tables 9 and 10 denote the aggregated values of the Tables 1 and 2 by LCCuPFWAAO.

Table 9: Aggregated Values of the Table 1 by LCCuPFAAO.

Alternatives	Aggregated Values
ζ_1	$\left(\begin{pmatrix}[s_{2.84}, s_{6.47}]e^{i[s_{2.78}, s_{6.62}]}, \\ [s_{2.46}, s_{6.2}]e^{i[s_{3.32}, s_{6.57}]}\end{pmatrix} \\ (s_{7.15}e^{is_{6.13}}, s_{6.46}e^{is_{6.53}})\right)$
ζ_2	$\left(\begin{pmatrix}[s_{4.56}, s_{10.15}]e^{i[s_{2.74}, s_{7.35}]}, \\ [s_{2.02}, s_{5.64}]e^{i[s_{3.28}, s_{6.76}]}\end{pmatrix} \\ (s_{5.91}e^{is_{9.2}}, s_{2.23}e^{is_{6.02}})\right)$
ζ_3	$\left(\begin{pmatrix}[s_{4.3}, s_{6.78}]e^{i[s_{3.4}, s_{7.86}]}, \\ [s_{1.86}, s_{7.7}]e^{i[s_{3.19}, s_{7.09}]}\end{pmatrix} \\ (s_{4.85}e^{is_{9.46}}, s_{3.94}e^{is_{5.63}})\right)$
ζ_4	$\left(\begin{pmatrix}[s_{4.14}, s_{7.94}]e^{i[s_{3.25}, s_{7.35}]}, \\ [s_{1.62}, s_{5.1}]e^{i[s_{3.19}, s_{6.63}]}\end{pmatrix} \\ (s_{6.61}e^{is_{10.2}}, s_{3.42}e^{is_{6.47}})\right)$

Table 10: Aggregated Values of the Table 1 by LCCuPFAAO.

Alternatives	Aggregated Values
ζ_1	$\left(\begin{pmatrix}[s_{4.34}, s_{6.42}]e^{i[s_{2.82}, s_{7.29}]}, \\ [s_{2.35}, s_{7.26}]e^{i[s_{2.29}, s_{6.45}]}\end{pmatrix} \\ (s_{7.14}e^{is_{7.36}}, s_{6}e^{is_{7.99}})\right)$
ζ_2	$\left(\begin{pmatrix}[s_{4.18}, s_{9.87}]e^{i[s_{3.06}, s_{7.44}]}, \\ [s_{3.42}, s_{6.27}]e^{i[s_{1.91}, s_{7.36}]}\end{pmatrix} \\ (s_{7.51}e^{is_{10.3}}, s_{3.51}e^{is_{4.29}})\right)$
ζ_3	$\left(\begin{pmatrix}[s_{4.48}, s_{9.03}]e^{i[s_{3.6}, s_{8.72}]}, \\ [s_{2.81}, s_{6.5}]e^{i[s_{2.35}, s_{6.81}]}\end{pmatrix} \\ (s_{6.48}e^{is_{6.71}}, s_{7.15}e^{is_{7.09}})\right)$
ζ_4	$\left(\begin{pmatrix}[s_{4.45}, s_{9.88}]e^{i[s_{2.56}, s_{7.07}]}, \\ [s_{1.55}, s_{4.84}]e^{i[s_{2.14}, s_{7.92}]}\end{pmatrix} \\ (s_{9.02}e^{is_{8.03}}, s_{4.58}e^{is_{5.43}})\right)$

Now, we compute the S.Fs of the aggregated information provided in Tables 9 & 10 to get a single numerical value for each alternative. After employing the S.F, we get Tables 11 & 12.

Table 11: Numerical Values of the Table 9 by LCCuPFWAAO.

Alternatives	Score Functions
ζ_1	$s_{9.9}$
ζ_2	$s_{9.96}$
ζ_3	$s_{9.93}$
ζ_4	$s_{9.95}$

Table 12: Numerical Values of the Table 10 by LCCuPFWAAO.

Alternatives	Score Functions
ζ_1	$s_{9.91}$
ζ_2	$s_{9.96}$
ζ_3	$s_{9.93}$
ζ_4	$s_{9.96}$

Here, we compute the score value for alternatives $\zeta_i; i = 1,2,3,4$. Table 13 represents the score values for alternatives $\zeta_i; i = 1,2,3,4$.

Table 13: Score Values by LCCuPFWAAO.

Alternatives	Score Values	Rank
ζ_1	$s_{0.7075}$	4
ζ_2	$s_{0.7114}$	1
ζ_3	$s_{0.7093}$	3
ζ_4	$s_{0.7111}$	2

The rank of the alternatives based on their score values by LCCuPFWAAO is given in Table 13. According to Table 13, the alternative ζ_2 has maximum score value. Thus, ζ_2 is the best option among the given alternatives.

6.5 Sensitivity Analysis

Here, we discuss the sensitivity analysis about the two approaches whose ranking values is given in Table 14.

Table 14: Ranking Values by LCCuPFAAO and LCCuPFWAAO.

Alternatives	LCCuPFAAO	LCCuPFWAAO
ζ_1	4	4
ζ_2	1	1
ζ_3	2	3
ζ_4	3	2

Table 14 indicates that the alternative ζ_2 is the best choice and ζ_1 is the worst choice based on both approaches. This means that the worst and best choices in the proposed approaches do not depend on whether attributes are given stakeholder-driven weighted or equally weights. The only instability exists in alternatives ζ_3 and ζ_4. The alternative ζ_3 decides better choice than ζ_4 under equal weighting but in case of stakeholder-driven weights, ζ_4 decides better choice than ζ_3.

7. Comparison Analysis

In this section, we offer a detailed comparison between the newly defined models and various existing approaches such as Alghazzawi approach (Alghazzawi et al., 2025), Malik approach (Malik et al., 2024), Alolaiyan approach (Alolaiyan et al., 2024), Kalsoom approach (Kalsoom et al., 2026), Xu approach (Xu et al., 2021), Qin approach (Qin et al., 2020), Garg approach (Garg, 2020), and Yasmin approach (Khan et al., 2024). Alghazzawi (Alghazzawi et al., 2025) and Malik et al. (Malik et al., 2024) proposed innovative DM approaches using LIFSs, Alolaiyan (Alolaiyan et al., 2024) and Kalsoom et al. (Kalsoom et al., 2026) employed LPFSs to establish some new DM techniques, Xu (Xu et al., 2021) and Qin et al. (Qin et al., 2020) developed new techniques using LIVIFSs, Garg et al. (Garg, 2020) used LIVPFSs to proposed a novel DM approach, and Yasmin et al. (Khan et al., 2024) discussed an innovative DM approach within the framework of LCIFSs. This comparison is investigated based on two key perspectives: ranking-wise performance and characteristic-wise evaluation. The ranking-wise performance contributes to evaluate the stability, consistency, and practical DM effectiveness of the proposed approaches. The characteristic-wise evaluation enables us to determine how the newly defined techniques differ from or enhance upon traditional formulations. By investigating both features, the analysis indicates the limitations, strengths, and distinguishing contributions of the defined approaches relative to existing approaches. Table 15 identifies ranking-wise performance between the newly defined models and existing approaches.

Table 15: Ranking-Wise Performance between the Newly Defined Models and Existing Approaches

Methods	Score Values	Ranking Values
(Alghazzawi et al., 2025)	0.0, 0.0, 0.0, 0.0	Not possible
(Malik et al., 2024)	0.0, 0.0, 0.0,0.0	Not possible
(Alolaiyan et al., 2024)	0.0, 0.0, 0.0,0.0	Not possible
(Kalsoom et al., 2026)	0.0, 0.0, 0.0,0.0	Not possible
(Xu et al., 2021)	0.0, 0.0, 0.0,0.0	Not possible
(Qin et al., 2020)	0.0, 0.0, 0.0,0.0	Not possible
(Garg, 2020)	0.0, 0.0, 0.0,0.0	Not possible
(Khan et al., 2024)	0.0, 0.0, 0.0,0.0	Not possible
Proposed Approach by LCCuPFAAO	$s_{0.724}, s_{0.727}, s_{0.726}, s_{0.725}$	$\zeta_2 \succ \zeta_3 \succ \zeta_4 \succ \zeta_1$
Proposed Approach by LCCuPFWAAO	$s_{0.7075}, s_{0.7114}, s_{0.7093}, s_{0.7111}$	$\zeta_2 \succ \zeta_4 \succ \zeta_3 \succ \zeta_1$

Table 15 identifies that the alternative ζ_2 is the best alternative and ζ_1 is the worst alternative by LCCuPFAAO and LCCuPFWAAO. Since, the proposed approaches involve four characteristics including linguistic feature, interval-valued MS and NMS degrees, complex-valued information and Pythagorean flexibility therefore the existing approaches cannot be employed for solving the given information. The existing DM approaches based on LIFSs, LPFSs, LIVIFSs, and LIVPFSs use real numbers or intervals and cannot capable to deal with complex-valued information. Moreover, the LCIFS model handle complex-valued uncertainty but they lack of addressing interval-valued uncertainty. As a result, all these approaches are limited to complex-valued or real-valued MS and NMS. They cannot evaluate the LCCuPFS information and hence, they are all provide zero score values.

Now, we investigate a characteristic-wise evaluation between the newly defined approaches and existing approaches. Table 16 shows characteristic-wise evaluation between them.

Table 16(a): Shows Characteristic-Wise Evaluation Between Proposed and Existing Approaches

Methods	Truth	Falsity	Linguistic Feature	Interval Support	Complex Interval Support
(Alghazzawi et al., 2025)	Yes	Yes	Yes	NO	No
(Malik et al., 2024)	Yes	Yes	Yes	No	No
(Alolaiyan et al., 2024)	Yes	Yes	Yes	No	No
(Kalsoom et al., 2026)	Yes	Yes	Yes	No	No

Table 16(b): Shows Characteristic-Wise Evaluation Between Proposed and Existing Approaches

Methods	Truth	Falsity	Linguistic Feature	Interval Support	Complex Interval Support
(Xu et al., 2021)	Yes	Yes	Yes	Yes	No
(Qin et al., 2020)	Yes	Yes	Yes	Yes	No
(Garg, 2020)	Yes	Yes	Yes	Yes	No
(Khan et al., 2024)	Yes	Yes	Yes	No	Yes
Proposed Approach by LCCuPFAAO	Yes	Yes	Yes	Yes	Yes
Proposed Approach by LCCuPFWAAO	Yes	Yes	Yes	Yes	Yes

Table 16 shows that the newly defined approaches improve the existing approaches by associating some additional information in their frameworks. The existing approaches have many limitations and constraints which are mentioned below:

1). Alghazzawi et al. (Alghazzawi et al., 2025) established some new AOs within the framework of LIFSs, and Malik et al. (Malik et al., 2024) demonstrated an innovative DM approach using LIFSs. The LIFS model is comprehensive for dealing with qualitative and uncertain information. But they cannot effectively present complex-valued and interval-valued information. Therefore, the approaches defined by Alghazzawi (Alghazzawi et al., 2025) and Malik (Malik et al., 2024) cannot be employed for solving the information given in Tables 1 & 2.

2). Alolaiyan et al. (Alolaiyan et al., 2024) presented new aggregation procedure for solving real-life problems using LPFSs, and Kalsoom et al. (Kalsoom et al., 2026) developed some Dombi AOs for LPFSs. LPFSs extend LIFSs by enabling Pythagorean flexibility in presenting qualitative and uncertain information. But the LPFS model cannot effectively solve complex-valued and interval-valued data. Therefore, the techniques demonstrated by Alolaiyan (Alolaiyan et al., 2024) and Kalsoom (Kalsoom et al., 2026) cannot be used for solving the data provided in Tables 1 & 2.

3). Xu et al. (Xu et al., 2021) established an innovative DM method within the framework of LIVIFSs, and Qin et al. (Qin et al., 2020) developed aggregation technique for DM problems based on LIVIFSs. LIVIFSs improve LIFSs by representing both MS and NMS degrees with interval-valued linguistic terms. But the LIVIFS framework cannot effectively handle complex-valued information. Therefore, the DM procedures proposed by Xu (Xu et al., 2021) and Qin (Qin et al., 2020) cannot be used for solving the information provided in Tables 1 & 2.

4). Garg (Garg, 2020) employed LIVPFSs to establish an innovative DM approach for solving real-world problems. LIVPFSs further improve LIFSs, LPFSs, and LIVIFSs by describing both MS and NMS degrees as interval-valued linguistic term with Pythagorean flexibility. But the LIVPFS framework cannot effectively handle complex-valued information. Therefore, the DM procedure demonstrated by Garg (Garg, 2020) cannot be used for solving the information provided in Tables 1 & 2.

5). Yasmin et al. (Khan et al., 2024) discussed an innovative DM technique based on LCIFSs. LCIFSs enhance the LIFS model by enabling the MS and NMS degrees to be described as complex-valued linguistic information. But the LCIFS framework cannot effectively handle interval-valued information. Therefore, the DM procedure demonstrated by Yasmin (Khan et al., 2024) cannot be used for solving the information provided in Tables 1 & 2.

The above discussion conclude that the newly defined approaches improve the existing approaches by associating some additional information including linguistic feature, interval-valued MS and NMS degrees, complex-valued information and Pythagorean flexibility. Therefore, the proposed approaches are more suitable for solving real-life problems.

8. Conclusion

In this paper, an innovative hybrid fuzzy model has been developed by combining CCuPFSs with linguistic characteristic to handle the limitations of existing fuzzy frameworks. The newly defined framework is capable of simultaneously addressing interval-valued imprecision, linguistic vagueness, and complex-valued cubic uncertainty, and hence contributing a more powerful and robust representation of daily real-life decision information. To facilitate practical application, some novel AOs have been proposed using the proposed approach. Moreover, the significance and effectiveness of the newly defined approaches were investigated through real-world financial investment DM problem, presenting its interpretability, robustness and superior capability in handling multifaceted uncertainty. The final results show that the newly defined model provides a reliable and comprehensive tool for complex DM applications and can be improved to other frameworks characterized by heterogeneous and uncertain information.

In the future, the proposed AOs can be extended by establishing some advance aggregation approaches such as Bonferroni, Hamacher, or Einstein AOs to further improve flexibility and interdependence of attributes in DM problems. The newly defined framework may also be combined with optimization or machine learning techniques to address dynamic and large-scale decision environments. Moreover, the proposed approaches may further be employed in some other domains such as sustainable energy planning, cybersecurity, risk assessment, and healthcare diagnosis.

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