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Modeling Multibody Systems Subjected to Impact with Friction

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ABSTRACT

Impact phenomena occur in a wide range of multibody systems, either accidentally or as part of their functional operations. Regardless of its serious degrading effects, it is challenging to model, especially when friction is considered, since the discontinuity due to friction is added to the discontinuity of the velocity due to impact. In this paper, an effective scheme applicable to multibody systems with motion constraints subjected to impacts with friction is developed. The main contribution of this study is the development of a numerical strategy to solve the nonlinear differential equations of motion for three-dimensional multibody systems experiencing impact with friction, as well as the analytical solution of these equations for planar multibody systems. Routh's incremental method is used to differentiate between the three possible modes during the contact period: continuous sliding, sticking and discontinuous sliding. A critical coefficient of friction is specified, which depends only on the system configuration, and it is used to determine whether the mode of contact is sliding or sticking. Three definitions for the coefficient of restitution are introduced to denote the end of impact. The proposed method is applied to a case study, and the dynamical behaviours are illustrated, with an emphasis on the effect of using each of the three coefficients of restitution. The results demonstrate that the present methodology is a powerful tool for modeling and simulating such practical problems and for assessing the use of different coefficients of restitution.

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1. Introduction

Modeling and simulating the dynamics of multibody systems often involve impact with friction, either as an unintended occurrence or an integral part of their functional operations. These contact-impact phenomena are prevalent across nearly all fields of engineering. Impacts can result from accidental collisions or be intentionally incorporated into various applications, such as forging machines, drilling devices, spot welding, surface writing, part assembly, tool manipulation, cooperative manipulators, and impulsive manipulation. During impact processes, sudden changes occur within the mechanical system over very short time intervals. This leads to high-magnitude forces, energy dissipation, and discontinuities in velocity and acceleration, among other challenges. Moreover, when impact occurs with friction, the discontinuity in Coulomb friction introduces additional nonlinearity, compounding the complexity already introduced by velocity discontinuities. Handling friction—even in planar cases—is a complex process. It involves two distinct modes: sticking and slipping, each requiring different approaches and solutions. For example, a rigid body sliding on a rough surface may encounter the Painlevé paradox, which remained unsolved until recently and continues to be an area of active research (Elkaranshawy, Mohamed, et al., 2017). In the dynamic formulation of multibody systems subjected to impact with friction, the system's configuration is assumed to remain unchanged. The coefficients of restitution and friction are used to account for impact plasticity and frictional effects, respectively. During planar impact, sliding may cease and persist until the end of the impact period, or it may restart in the opposite direction. As a result, conventional friction force laws cannot be directly applied to friction impulses computed over the impact duration. Routh (Routh, 1897) recognized this challenge and proposed an incremental method to distinguish between different contact modes. However, many classical dynamics textbooks (Kane & Levinson, 1985; Whittaker, 1964) still use a simplified approach, assuming that during impact, the contact point either continuously slides in one direction or remains stationary. Kane and Levinson (Kane & Levinson, 1985) later pointed out that this classical approach could lead to energetic inconsistencies when sliding reverses direction during planar collisions. To address this paradox, Keller (KELLER, 1986) revisited Routh's incremental method. However, incorporating Newton's coefficient of restitution within Routh's framework failed to resolve the inconsistency, leading to the proposal of Poisson's coefficient of restitution. Meanwhile, Stronge (Stronge, 1990) argued that even the Poisson–Routh model could be energetically inconsistent and instead introduced an energetic coefficient of restitution as a more consistent alternative. Wang and Mason, as well as Ahmed et al. (AHMED et al., 1999; Wang & Mason, 1992), presented a formulation for impact with friction using the Routh graphical method with Poisson's coefficient of restitution. Charles et al. (Charles et al., 2018) extended a time discretisation numerical method for nonsmooth contact dynamics of a frictionless evolution problem to a time-stepping scheme for multibody dynamics with impact and friction. Aghili 2020) developed a combined slipping and sticking model for multibody systems subjected to impacts with friction. He specified upper-bound limits for the coefficient of restitution and the coefficient of friction which led to an energetically consistent model. Elkaranshawy and Bajaba (Elkaranshawy & Bajaba, 2024) formulated the nonlinear equations of motion using the normal impulse at the contact point as a time-independent variable. They derived analytical algebraic solutions to evaluate the impact variables. In their review paper, Corral et al. (2021) provided a resource for beginning researchers to choose the appropriate contact model for their work by describing different existing approaches and their most distinctive features. During three-dimensional impact, the contact point may continuously change its sliding direction, come to a complete stop and remain stuck, or experience an instantaneous stop followed by sliding in a new direction without deviation. The equations of motion governing these scenarios form a system of highly nonlinear ordinary differential equations. In most cases, exact solutions are not feasible, making numerical methods the primary approach for solving them. Hence, several researchers (Bhatt and Koechling (V Bhatt & J Koechling, 1995; V Bhatt & Jeff Koechling, 1995), Batlle (Batlle, 1996), and Elkaranshawy et al. (Elkaranshawy et al., 2015; Elkaranshawy, Abdelrazek, et al., 2017), Chatterjee et al. (Chatterjee & Bowling, 2018)) have investigated and analyzed the system of nonlinear differential equations, which represents the tangential velocity during impact with friction. Numerous other researchers (Stronge (Stronge, 1994), Batlle and Cardona (Batlle & Cardona, 1998), Zhen and Liu (Zhen & Liu, 2007), Elkaranshawy (Elkaranshawy, 2007; Elkaranshawy, 2011), Uchida et al. (Uchida et al., 2015), and Jia and Wang (Jia & Wang, 2017)) have considered the general problem of three-dimensional impact with friction. Zhang and Sharf (ZHANG & SHARF, 2007) developed an integrated form of Keller's equations to solve for post-impact velocities. Pandrea and Stanescu (Pandrea & Stănescu, 2019) employed the concept of inertance, derived from the theory of screws, to investigate the impact of friction on one or two rigid bodies without constraints. In their article, Peng et al. (Peng et al., 2020) employed D'Alembert's principle to derive differential algebraic equations for a nonsmooth multibody system in a symplectic discrete format. Passas and Natsiavas (Passas & Natsiavas, 2022) developed a time-stepping scheme applicable to mechanical systems subject to bilateral and unilateral motion constraints in the presence of impact with friction, even when contact exists for a period after an impact. Recently, Flores (Flores, 2022) and Flores et al. (Flores et al., 2023) presented comprehensive discussions and reviews of the state-of-the-art modelling of frictional contacts and impacts in multibody dynamics in two review papers. The Routh incremental method is not only physically accurate but also essential for preventing misleading results and ensuring realistic or reliable simulated motion. Despite its significance, it has not been widely adopted in many studies on modeling impact with friction. Elkaranshawy and Bajaba (Elkaranshawy & Bajaba, 2024) applied the Routh method in the planar case; therefore, this paper extends its application to three-dimensional systems. A precise and effective formulation is developed for modeling and simulating impact with friction in general three-dimensional multibody systems. Assuming a single contact point and Coulomb's friction, the nonlinear equations of motion are formulated using the normal impulse at the contact point as a time-like independent variable. The derived equations reveal that, in general, sliding continuously changes direction. Sliding may come to a stop and transition into a persistent sticking phase, or it may resume in a new direction. To accurately capture these transitions, Routh's incremental method is applied to differentiate between sliding and sticking behaviors. A critical coefficient of friction—dependent solely on the system configuration—is identified and used to determine whether the contact mode is sliding or sticking. Once the sticking mode is reached during impact, it will persist only if the static coefficient of friction exceeds the critical coefficient of friction. Additionally, a coefficient of restitution is introduced to account for impact plasticity, considering the three widely used definitions: Newton's, Poisson's, and the energetic coefficient of restitution. Finally, a numerical strategy is proposed for solving the nonlinear equations of motion governing three-dimensional systems. All potential sliding and sticking situations are taken into consideration by the strategy. Additionally, a thorough analysis is conducted for the planar systems. The several contact modes of impact are distinguished in this analysis, and each mode has an analytical algebraic solution. To demonstrate the efficacy of the formulation and to support the algebraic solutions, Kane's example is considered and carefully investigated. The results of the three definitions of the coefficient of restitution are also examined using it.

2. Materials and Methods

2.1 Equations of Motion

According to Goldsmith (Goldsmith, 2001), the generalized impulse H During single-point impact with friction, it can be expressed as:

$$Cap\Delta(\frac{\partial T}{\partial \dot{q}}) = \mathrm{H} \tag{1}$$

where T represents the total kinetic energy and $\dot{q}$ denotes the n-dimensional generalized joint velocities. The total kinetic energy is given by:

$$T = \frac{1}{2}\dot{q}^{T} M \dot{q} \tag{2}$$

and the generalized impulse can be written as:

$$\mathrm{H} = J^{T}\mathrm{I} \tag{3}$$

where $M(\mathrm{q})$ is the inertia matrix of the multibody system, with $M(.) \in R^{n\times n}$, q represents the n-dimensional generalized joint coordinates, I is the impulse at the impact point, and $J(\mathrm{q})$ is the Jacobian matrix that transforms generalized joint velocities into the velocity of the impact point as:

$$\mathrm{v} = J\dot{q} \tag{4}$$

$$\mathrm{v} = [v_n \quad \mathrm{v_t}]^{T} \tag{5}$$

$$J = [\alpha^{T} \quad \beta^{T}]^{T} \tag{6}$$

where v_n and $\mathrm{v_t}$ are the standard and tangential components of the velocity of the impact point, respectively, and $v_n \in R^{1\times 1}$, $\mathrm{v_t} \in R^{2\times 1}$, $\alpha \in R^{n\times 1}$, $\beta \in R^{n\times 2}$. I is defined as

$$\mathrm{I} = [I_n \quad \mathrm{I_t}]^{T} \tag{7}$$

where I_n and $\mathrm{I_t}$ are the standard and tangential components of that impulse, respectively; $I_n \in R^{1\times 1}$ and $\mathrm{I_t} \in R^{2\times 1}$. Hence, Eq (1) yields

$$\Delta\dot{q} = M^{-1}J^{T}\mathrm{I} \tag{8}$$

The differential form of this equation is:

$$\frac{d\dot{q}}{dt} = M^{-1}J^{T}\frac{d\mathrm{I}}{dt} \tag{9}$$

Equations (4) and (9) gives:

$$\frac{d\mathrm{v}}{dt} = D\frac{d\mathrm{I}}{dt} \tag{10}$$

where is the Jacobian inertia $D \in R^{3\times 3}$ which depends only upon system configuration and is a symmetric positive definite matrix given by:

$$D = JM^{-1}J^{T} = \begin{bmatrix} a & \mathrm{c}^{T} \\ \mathrm{c} & b \end{bmatrix} \tag{11}$$

with

$$a = \alpha^T M^{-1}\alpha, \qquad b = \beta^T M^{-1}\beta, \qquad c = \beta^T M^{-1}\alpha, \tag{12}$$

and $a \in R^{1\times1}$, $b \in R^{2\times2}$, $c \in R^{2\times1}$.

2.2 Coulomb Friction Law and Contact Modes

Coulomb's friction law with stiction and infinite tangential stiffness at the impact point are assumed. As a result, three distinct contact modes can be identified:

2.2.1 Sliding Mode

In the sliding mode, the following relationship is used:

$$\frac{dI_t}{dt} = -\mu_D \sigma \frac{dI_n}{dt} \tag{13}$$

where μ_D is the kinetic coefficient of friction and σ is the unit vector defining the sliding direction, given by:

$$\sigma = \frac{v_t}{\|v_t\|} \tag{14}$$

Thus, we can express:

$$\frac{dI_t}{dI_n} = -\mu_D \sigma \tag{15}$$

The normal impulse I_n is chosen as the impact parameter, as it acts as a time-like variable that starts at zero at the onset of impact and increases continuously throughout the impact. Substituting Eq (15) in Eq (9) yields

$$\frac{d\dot{q}^s}{dI_n} = M^{-1}J^T[1 \quad -\mu_D \sigma]^T = L^s \tag{16}$$

In this paper, superscript (s) is used to denote the sliding mode, while (ns) is used for the non-sliding (sticking) mode. Substituting Eq (15) in Eq (10) results in:

$$\frac{d{v_n}^s}{dI_n} = a - \mu_D c^T \sigma = \frac{1}{\zeta^s} \tag{17}$$

$$\frac{dv_t}{dI_n} = c - \mu_D b \sigma = \frac{1}{\epsilon} \tag{18}$$

Equation (17) indicates that v_n depends on the sliding direction σ, which in turn is determined by v_t. Conversely, Eq (18) shows that v_t is independent of v_n. Consequently, to solve for v_t, there is no need to solve for v_n first. The opposite is not true, to solve for v_n one should solve for v_t first. Hence, once the solution for v_t is obtained, the solution for v_n can directly be obtained. The assessment of this system of highly nonlinear differential equations depends mainly on the tangential velocity. Equation (18) demonstrates that the evolution of the tangential velocity, $\frac{dv_t}{dI_n}$, generally follows a direction that does not align with the direction of the tangential velocity v_t itself. This implies that during the collision interval, the tangential velocity may continuously change direction, meaning that σ is not a constant vector.

2.2.2 Sticking Mode

In the sticking mode, the following conditions hold:

$$v_t = 0, \ \frac{dv_t}{dt} = 0 \tag{19}$$

By substituting Eq (5) and Eq (7) in Eq (10) and applying Eq (19), we obtain:

$$\frac{dI_t}{dI_n} = -b^{-1}c \ \text{ or } \ \frac{dI_t}{dt} = -b^{-1}c\frac{dI_n}{dt} \tag{20}$$

A critical coefficient of friction can be introduced as:

$$\mu_c = \|-b^{-1}c\| \tag{21}$$

However, according to Coulomb friction law during non-sliding

$$\left\|\frac{dI_t}{dt}\right\| \leq \mu_s \frac{dI_n}{dt} \tag{22}$$

where μ_s is the static coefficient of friction. For sticking mode to persist once it has been established, the following condition must be met:

$$\mu_s \geq \mu_c \tag{23}$$

Otherwise, the available friction would be insufficient to maintain the non-sliding state, and sliding would resume in a new direction. Substituting Eq (20) into Eq (9) gives:

$$\frac{d\dot{q}^{ns}}{dI_n} = M^{-1}J^T[1 \quad -b^{-1}c]^T = L^{ns} \tag{24}$$

Substituting Eq (20) in Eq (10) results in:

$$\frac{d{v_n}^{ns}}{dI_n} = a - c^T b^{-1} c = \frac{1}{\zeta^{ns}} \tag{25}$$

2.2.3 Discontinuous Mode

For the discontinuous mode, the equations provided in Section 3.1 remain valid up until the sliding velocity vanishes. In this case, the condition:

$$\mu_s \leq \mu_c \tag{26}$$

indicates that friction is insufficient to sustain the non-sliding mode, leading to a restart of sliding in a new direction. Since the sliding velocity, in this case, restarts from zero, it will follow the same direction as $\frac{dv_t}{dI_n}$. By applying Eq (18), the following relationship can be established:

$$\sigma_F{}^T R^T(c - \mu_D b \sigma_F) = 0 \tag{27}$$

where σ_F is the unit vector defining the new sliding direction, and R is the transformation matrix, given by:

$$R = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \tag{28}$$

Simultaneously

$$\sigma_F{}^T \sigma_F = 1 \tag{29}$$

Equations (27) and (29) can be solved to specify the new sliding direction. Since more than one solution can be obtained from these two equations, the solution which produces a centrifugal direction, which satisfy the relation $\frac{dv_t}{dI_n} = +\frac{dv_t}{dI_n}\sigma_F$The correct one is the one that restarts from zero and increases.

2.3 Restitution Law

The deformation of an object during impact consists of two distinct phases: *compression* and *restitution*. At the end of the compression phase, the standard velocity component v_n becomes zero, marking the start of the restitution phase, which continues until the impact is fully resolved. The coefficient of restitution determines the completion of effects. There are three commonly used definitions for the coefficient of restitution:

- Newton's coefficient

$$e = -\frac{v_{ne}}{v_{ni}} \tag{30}$$

where v_{ni} and v_{ne} represent the standard velocity components at the beginning and end of impact, respectively.

- Poisson's coefficient

$$e = \frac{I_{ne} - I_{nc}}{I_{nc}} \tag{31}$$

where I_{nc} and I_{ne} are the normal impulses at the end of the compression phase and the end of impact, respectively.

- energetic coefficient

$$e^2 = -\frac{W_{nR}}{W_{nc}} = -\frac{W_{ne} - W_{nc}}{W_{nc}} \tag{32}$$

where W_{ne} (with $W_n \leq 0$), W_{nc} (with $W_{nc} \leq 0$) and W_{nR} (with $W_{nR} \geq 0$) denote the work done by the standard component of the reaction force during the total impact period, the compression phase, and the restitution phase, respectively. By integrating Eq (17) and Eq (25), we obtain:

$$I_{nc} = \int_{v_{ni}}^{0} \zeta \; \mathrm{d}v_n \tag{33}$$

$$I_{ne} = \int_{v_{ni}}^{v_{ne}} \zeta \; \mathrm{d}v_n \tag{34}$$

The work done by the standard component of the reaction force is given by:

$$W_n = \int_0^{I_n} v_n \, \mathrm{d}I_n, \qquad W_{ne} = \int_0^{I_{ne}} v_n \, \mathrm{d}I_n \tag{35}$$

$$W_{nc} = \int_0^{I_{nc}} v_n \, \mathrm{d}I_n = \int_{v_{ni}}^{0} v_n \, \zeta \, \mathrm{d}v_n, \qquad W_{nR} = \int_{I_{nc}}^{I_{ne}} v_n \, \mathrm{d}I_n = \int_0^{v_{ne}} v_n \, \zeta \, \mathrm{d}v_n \tag{36}$$

where W_n ($W_n \leq 0$) is the work done by the the standard component of reaction force at any instant during impact, and ζ could be ζ^s (sliding) or ζ^{ns}(sticking). Additionally, friction dissipates energy only during sliding but not during sticking. The work dissipated by friction is given by:

$$W_t = \int_0^{\mathrm{I_t}} \mathrm{v_t} \, . \, \mathrm{dI_t}, \qquad W_{te} = \int_0^{\mathrm{I_{te}}} \mathrm{v_t} \, . \, \mathrm{dI_t}, \tag{37}$$

where $\mathrm{I_t}, \mathrm{I_{te}}$ denote the tangential impulse at any given instant and the end of impact, respectively, while W_t $(W_t \leq 0)$ and W_{te} $(W_{te} \leq 0)$ represent the work done by friction at any instant and throughout the impact period, respectively. As discussed in Section 3.1, the sliding direction of the impact point in three-dimensional multibody systems continuously shifts during the impact period. Generally, solving the bidimensional nonlinear differential equations of motion (Eq (17) and Eq (18)) requires numerical integration, as demonstrated in previous studie (Elkaranshawy et al., 2015; Elkaranshawy, Abdelrazek, et al., 2017).

However, if the sliding direction is confined to a discrete set of predefined *invariant directions*, algebraic solutions are possible (Elkaranshawy (Elkaranshawy, 2021)). Figure 1 illustrates the numerical process for solving the impact-with-friction problem in a three-dimensional multibody system. For planar impacts, algebraic solutions are also feasible since parameters such as b, c, $\mathrm{v_t}$, σ, becomes scalar quantities. For instance, σ can be either -1 or 1. Thus, the nonlinear differential equations, Eqs. (15)–(18), and the integrals in Eqs. (33)–(36) can be solved analytically, as discussed in the next section.

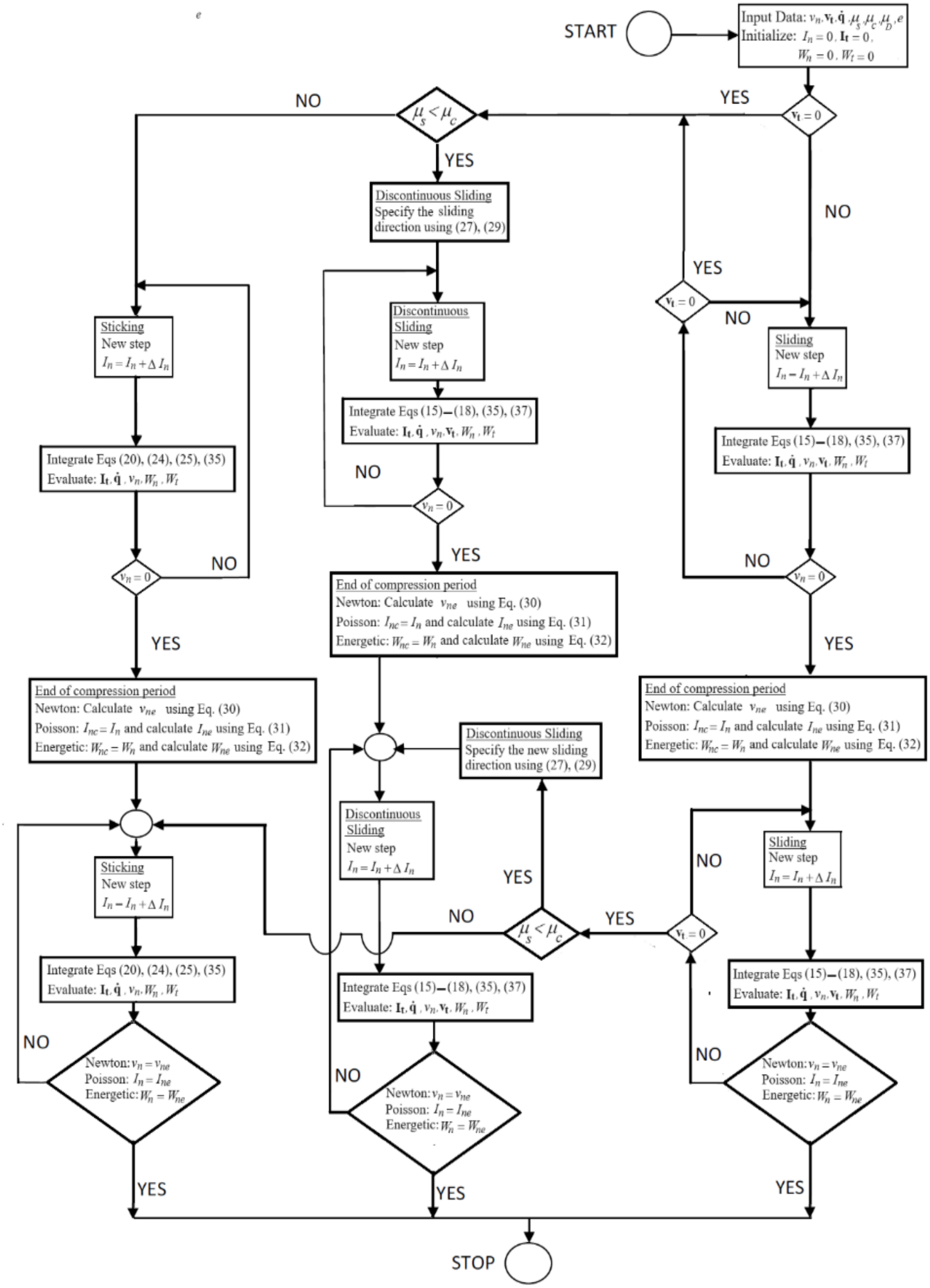

Figure 1: Numerical Procedure for the Solution of the Three-Dimensional Systems

2.4 Planar Impact

2.4.1 Types of Impact

Consider a planar impact scenario where the motion starts with sliding in a specific direction, denoted by $\sigma = \sigma_I$. If the tangential velocity reaches zero during the impact period, the corresponding normal impulse I_{nh} and the normal velocity v_{nh} At this instant, the value can be determined by integrating Eqs. (17) and (18). During this phase, the tangential velocity remains in the original direction σ_I until it vanishes. Consequently $\zeta^s = \zeta_I^s$ remains constant, leading to:

$$I_{nh} = -\frac{\|v_{ti}\|}{\sigma_I^T(\mathrm{c} - \mu_D b\sigma_I)} \tag{38}$$

$$v_{nh} = v_{ni} + \frac{I_{nh}}{\zeta_I^s} \tag{39}$$

If the tangential velocity persists throughout the entire compression phase, the normal impulse at this instant, I_{np}, is obtained by integrating Eq (17):

$$I_{np} = -\zeta_I^s v_{ni} \tag{40}$$

Similarly, if the tangential velocity remains nonzero until the end of impact, the normal impulse at this instant, I_{nf}, is given by:

$$I_{nf} = \zeta_I^s(v_{nf} - v_{ni}) \tag{41}$$

2.4.1.1 Sliding vs. Sticking Criteria

From the above expressions, the conditions that determine whether sliding or sticking occurs during the impact can be summarized as follows:

- If $I_{nh} > I_{nf}$ sliding continues until the end of impact.
- If $I_{nh} < I_{np}$ sliding stops during the compression phase.
- If $I_{np} < I_{nh} < I_{nf}$ sliding stops during the restitution phase.

The values of I_{np}, I_{nh}, and I_{nf} depend on:

- The inertia and orientation of the multibody system
- The initial velocity
- The coefficient of restitution

If sliding persists throughout the impact period, all three definitions of the coefficient of restitution (Newtonian, Poisson, and energetic) become identical since ζ_I^s remains constant. In this case, the normal velocity at the end of impact v_{nf} can be determined using Eq (30), Eq (31), or Eq (32):

$$v_{nf} = -ev_{ni} \tag{42}$$

2.4.1.2 Transition from Sliding to Sticking

If the initial sliding in the direction σ_I stops during the contact period, the motion transitions into a sticking mode, which will persist if ($\mu_s \geq \mu_c$). If this condition is not met ($\mu_s < \mu_c$), friction is insufficient to maintain the sticking mode, and the sliding restarts in a new direction σ_F. This results in a reversal of sliding, expressed mathematically as:

$$\sigma_I\sigma_F = -1 \tag{43}$$

2.4.1.3 Types of Motion

For planar impact, the five possible motion types resulting from different impact scenarios are summarised in Table 1.

Table 1: Contact-Impact Types

	$\mu_s < \mu_c$	$\mu_s > \mu_c$
$I_{nh} > I_{nf}$	Permanent Sliding	
$I_{nh} < I_{np}$	Reverse Sliding in Compression	Non-Sliding in Compression
$I_{np} < I_{nh} < I_{nf}$	Reverse Sliding in Restitution	Non-Sliding in Restitution

2.4.2 Algebraic Solutions

If the tangential velocity becomes zero during the compression phase, Eq (33) - Eq (36) are utilized to determine the normal impulses at the end of the compression period as

$$I_{nc} = \zeta_I(v_{nh} - v_{ni}) - \zeta_F v_{nh} \tag{44}$$

the overall impact is:

$$I_{ne} = \zeta_I(v_{nh} - v_{ni}) + \zeta_F(v_{ne} - v_{nh}) \tag{45}$$

The work done by the standard component of the reaction force during compression is:

$$W_{nc} = \frac{1}{2}\zeta_I(v_{nh}^2 - v_{ni}^2) - \frac{1}{2}\zeta_F v_{nh}^2 \tag{46}$$

and the work done by the standard component of the reaction force during restitution is:

$$W_{nR} = \frac{1}{2}\zeta_F v_{ne}^2 \tag{47}$$

If the tangential velocity becomes zero in restitution phase, the mentioned quantities can be determined from Eq (33) - Eq (36) as

$$I_{nc} = -\zeta_I v_{ni} \tag{48}$$

$$I_{ne} = \zeta_I(v_{nh} - v_{ni}) + \zeta_F(v_{ne} - v_{nh}) \tag{49}$$

$$W_{nc} = -\frac{1}{2}\zeta_I v_{ni}^2 \tag{50}$$

$$W_{nR} = \frac{1}{2}\zeta_I v_{nh}^2 + \frac{1}{2}\zeta_F(v_{ne}^2 - v_{nh}^2) \tag{51}$$

Utilizing Eq (15), Eq (18), and Eq (37), the work done by the frictional component of impact can be determined as

$$W_{te} = \frac{1}{2}\mu_D\sigma_I\epsilon_I v_{ti}^2 + \frac{1}{2}\mu_D\sigma_F\epsilon_F v_{te}^2 \tag{52}$$

The following quantities are identified to for the planar impact:

$$\zeta_I^s = (a - \mu_D c\sigma_I)^{-1}, \quad \zeta_F^s = (a - \mu_D c\sigma_F)^{-1}, \quad \zeta_F^{ns} = (a - b^{-1}c^2)^{-1} \tag{53}$$

$$\epsilon_I^s = (c - \mu_D b\sigma_I)^{-1}, \quad \epsilon_F^s = (c - \mu_D b\sigma_F)^{-1}, \quad \epsilon_F^{ns} = 0 \tag{54}$$

$$\mathrm{L}_I^s = M^{-1}J^T[1 \quad -\mu_D\sigma_I]^T, \quad \mathrm{L}_F^s = M^{-1}J^T[1 \quad -\mu_D\sigma_F]^T, \quad \mathrm{L}_F^{ns} = M^{-1}J^T[1 \quad -b^{-1}c]^T \tag{55}$$

Hence, the following are the algebraic solutions for the equations of motion for the five types of planar impact. It should be noted that the forms for all effect are the same for the three definitions of the coefficient of restitution accept the standard component of velocity.

2.4.2.1 Permanent Sliding

Newton's, Poisson's, and energetic coefficients:

$$v_{ne} = -ev_{ni} \tag{56}$$

$$v_{te} = v_{ti} + I_{ne}(c - \mu_D b\sigma) \tag{57}$$

$$I_{ne} = \zeta(v_{ne} - v_{ni}) \tag{58}$$

$$I_{te} = -\mu_D\sigma I_{ne} \tag{59}$$

$$\Delta\dot{\mathrm{q}} = I_{ne}\mathrm{L}^s \tag{60}$$

2.4.2.2 Reverse Sliding in Compression

Newton's coefficient: $$v_{ne} = -ev_{ni} \tag{61}$$

Poisson's coefficient: $$v_{ne} = -e\left(v_{ni}\left(\frac{\zeta_I^s}{\zeta_F^s}\right) + v_{nh}\left(1 - \frac{\zeta_I^s}{\zeta_F^s}\right)\right) \tag{62}$$

Energetic coefficient: $$v_{ne} = -e\sigma_n\sqrt{v_{ni}^2\left(\frac{\zeta_I^s}{\zeta_F^s}\right) + v_{nh}^2\left(1 - \frac{\zeta_I^s}{\zeta_F^s}\right)} \tag{63}$$

$$v_{te} = v_{ti} + I_{nh}(c - \mu_D b\sigma_I) + (I_{ne} - I_{nh})(c - \mu_D b\sigma_F) \tag{64}$$

$$I_{ne} = \zeta_I^s(v_{nh} - v_{ni}) + \zeta_F^s(v_{ne} - v_{nh}) \tag{65}$$

$$I_{te} = -\mu_D\sigma_I I_{nh} - \mu_D\sigma_F(I_{ne} - I_{nh}) \tag{66}$$

$$\Delta\dot{\mathrm{q}} = I_{nh}\mathrm{L}_\mathrm{I}^\mathrm{s} + (I_{ne} - I_{nh})\mathrm{L}_\mathrm{F}^\mathrm{s} \tag{67}$$

where σ_n is the sign of v_n, which could be 1 or -1 and given by:

$$\sigma_n = \frac{v_n}{\|v_n\|} \tag{68}$$

2.4.2.3 Non-sliding in Compression

Newton's coefficient: $$v_{ne} = -ev_{ni} \tag{69}$$

Poisson's coefficient: $$v_{ne} = -e\left(v_{ni}\left(\frac{\zeta_I^s}{\zeta_F^{ns}}\right) + v_{nh}\left(1 - \frac{\zeta_I^s}{\zeta_F^{ns}}\right)\right) \tag{70}$$

Energetic coefficient: $$v_{ne} = -e\sigma_n\sqrt{v_{ni}^2\left(\frac{\zeta_I^s}{\zeta_F^{ns}}\right) + v_{nh}^2\left(1 - \frac{\zeta_I^s}{\zeta_F^{ns}}\right)} \tag{71}$$

$$v_{te} = 0 \tag{72}$$

$$I_{ne} = \zeta_I^s(v_{nh} - v_{ni}) + \zeta_F^{ns}(v_{ne} - v_{nh}) \tag{73}$$

$$I_{te} = -\mu_D\sigma_I I_{nh} - b^{-1}c(I_{ne} - I_{nh}) \tag{74}$$

$$\Delta\dot{\mathrm{q}} = I_{nh}\mathrm{L}_\mathrm{I}^\mathrm{s} + (I_{ne} - I_{nh})\mathrm{L}_\mathrm{F}^\mathrm{ns} \tag{75}$$

2.4.2.4 Reverse Sliding in Restitution

Newton's coefficient: $$v_{ne} = -ev_{ni} \tag{76}$$

Poisson's coefficient: $$v_{ne} = -ev_{ni}\left(\frac{\zeta_I^s}{\zeta_F^s}\right) + v_{nh}\left(1 - \frac{\zeta_I^s}{\zeta_F^s}\right) \tag{77}$$

Energetic coefficient: $$v_{ne} = -\sigma_n\sqrt{e^2 v_{ni}^2\left(\frac{\zeta_I^s}{\zeta_F^s}\right) + v_{nh}^2\left(1 - \frac{\zeta_I^s}{\zeta_F^s}\right)} \tag{78}$$

$$v_{te} = v_{ti} + I_{nh}(c - \mu_D b\sigma_I) + (I_{ne} - I_{nh})(c - \mu_D b\sigma_F) \tag{79}$$

$$I_{ne} = \zeta_I^s(v_{nh} - v_{ni}) + \zeta_F^s(v_{ne} - v_{nh}) \tag{80}$$

$$I_{te} = -\mu_D\sigma_I I_{nh} - \mu_D\sigma_F(I_{ne} - I_{nh}) \tag{81}$$

$$\Delta\dot{\mathrm{q}} = I_{nh}\mathrm{L}_\mathrm{I}^\mathrm{s} + (I_{ne} - I_{nh})\mathrm{L}_\mathrm{F}^\mathrm{s} \tag{82}$$

2.4.2.5 Non-Sliding in Restitution

Newton's coefficient: $$v_{ne} = -e v_{ni} \tag{83}$$

Poisson's coefficient: $$v_{ne} = -e v_{ni}\left(\frac{\zeta_I^s}{\zeta_F^{ns}}\right) + v_{nh}\left(1 - \frac{\zeta_I^s}{\zeta_F^{ns}}\right) \tag{84}$$

Energetic coefficient: $$v_{ne} = -\sigma_n \sqrt{e^2 v_{ni}^2\left(\frac{\zeta_I^s}{\zeta_F^{ns}}\right) + v_{nh}^2\left(1 - \frac{\zeta_I^s}{\zeta_F^{ns}}\right)} \tag{85}$$

$$I_{ne} = \zeta_I^s(v_{nh} - v_{ni}) + \zeta_F^{ns}(v_{ne} - v_{nh}) \tag{86}$$

$$v_{te} = 0 \tag{87}$$

$$I_{te} = -\mu_D \sigma_I I_{nh} - b^{-1} c (I_{ne} - I_{nh}) \tag{88}$$

$$\Delta\dot{\mathrm{q}} = I_{nh}\mathrm{L_I^s} + (I_{ne} - I_{nh})\mathrm{L_F^{ns}} \tag{89}$$

3. Numerical Simulation

The following is the well know Kan's problem (Kane & Levinson, 1985): The free end B of a double pendulum strikes the ground while the angular velocity of rod OA is 0.1 rad/s and the angular velocity of rod AB is 0.2 rad/s as shown in Fig. 2. Each of the two identical and uniform rods has a mass of 3 kg and a length of 2 m. Four cases were solved for the original problem. The method used produced a kinetic energy increase in two cases. In the following, one of them will be solved with the methods introduced in this paper and the results will be compared. The coefficient of restitution is assumed to be $e = 0.7$, and the coefficients of static and dynamic friction are $\mu_s = 0.51, \mu_D = 0.5$.

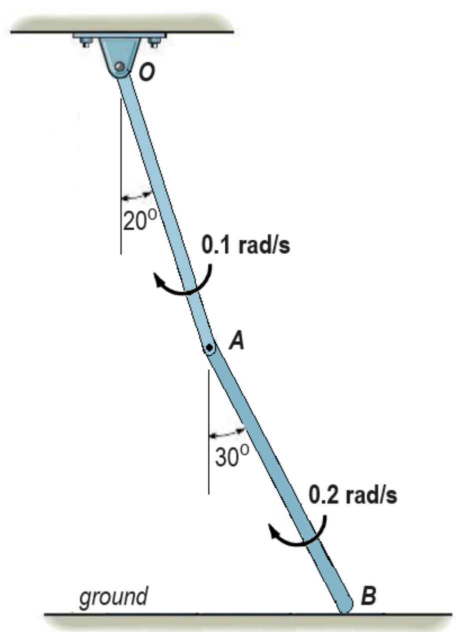

Figure 2: Impact between a Double Pendulum and a Frictional Ground Floor

The initial generalized joint velocities, mass matrix, the Jacobian matrix, and the Jacobian inertia matrix are given by:

$\dot{q}_i = [-0.1 \quad -0.2]^T$ rad/s, $M = \begin{bmatrix} 16 & 5.90885 \\ 5.90885 & 4 \end{bmatrix}$,

$J = \begin{bmatrix} 0.684 & 1 \\ 1.8794 & 1.7321 \end{bmatrix}$, $D = \begin{bmatrix} 0.33652 & 0.507103 \\ 0.507103 & 0.813446 \end{bmatrix}$

The initial normal velocity, the initial tangential velocity, and the normal velocity v_{nh} if the tangential velocity vanishes during impact are:

$v_{in} = -0.2684$ m/s, $v_{it} = -0.534349$ m/s, $v_{nh} = 0.0766332$ m/s

The critical coefficient of friction is: $\mu_c = 0.6234$, the normal impulse I_{nh} if the tangential velocity vanishes during impact, the normal impulse I_{np} if the tangential velocity does not vanish up to the end of compression phase, and the normal impulse I_{nf} if the tangential velocity does not vanish up to the end of impact are:

$I_{nh} = 0.584738$ N.s, $I_{np} = 0.454867$ N.s, and $I_{nf} = 0.773274$ N.s

Hence

$I_{np} < I_{nh} < I_{nf}$ and $\mu_s < \mu_c$

Therefore, the type of impact is *reverse in restitution* and the parameters ζ_I^s, ζ_F^s, L_I^s, and L_F^s are

$\zeta_I^s = 1.69471$, $\zeta_F^s = 12.0527$

$L_I^s = [-0.155786 \quad 0.696636]^T$, and $L_F^s = [-0.0623762 \quad 0.125636]^T$

The problem was solved in (Kane & Levinson, 1985) using the traditional method used in textbooks like (Whittaker, 1964). In the classical method, the motion during impact is assumed to be either continuously sticking or continuously sliding in the original sliding direction, which is incorrect. The sliding reverses its direction during the impact period. To solve the problem with the classical method, we start with assuming sticking, the obtained solution does not fulfill the conditions that $\frac{I_{te}}{I_{te}} \leq \mu_s$. Hence, the sliding is deduced, and the obtained results are:

$v_{ne} = 0.1879$ m/s, $v_{te} = 0.0177$ m/s,

$I_{ne} = 5.4995$ N.s, $I_{te} = -2.7498$ N.s,

$\Delta\dot{q} = [-0.3430 \quad 0.6909]^T$ rad/s, $\dot{q}_e = [-0.4430 \quad 0.4909]^T$ rad/s,

$\Delta T = 0.4889$ N.m

where $\Delta T = T_e - T_i$ is the change in the kinetic energy of the system which can be calculated from Eq (2). A positive ΔT indicates that the classical method leads to an increase in energy, which is physically unacceptable. To solve this energetic inconsistency, Routh's method was revisited, and all three definitions of the coefficient of restitution were examined. The resulting impact variables depend on the final velocity, which varies across the three restitution definitions, hence:

For the Newton's coefficient of restitution

$v_{ne} = 0.1879$ m/s, $v_{te} = 0.1346$ m/s,

$I_{ne} = 1.9256$ N.s, $I_{te} = -0.3781$ N.s,

$\Delta\dot{q} = [-0.1747 \quad 0.5758]^T$ rad/s, $\dot{q}_e = [-0.2747 \quad 0.3758]^T$ rad/s,

$W_{ne} = 0.1213$ N.m, $\Delta T = -0.00196$ N.m

The positive W_n presents discrepancy.

For the Poisson's coefficient of restitution

$v_{ne} = 0.0923$ m/s, $v_{te} = 0.0189$ m/s,

$I_{ne} = 0.7733$ N.s, $I_{te} = 0.1981$ N.s,

$\Delta\dot{q} = [-0.1029 \quad 0.4310]^T$ rad/s, $\dot{q}_e = [-0.2029 \quad 0.2310]^T$ rad/s,

$W_{ne} = -0.0401$ N. m, $\Delta T = -0.1192$ N. m

For the energetic coefficient of restitution

$v_{ne} = 0.1001$ m/s, $v_{te} = 0.0283$ m/s,

$I_{ne} = 0.8670$ N. s, $I_{te} = 0.1512$ N. s,

$\Delta\dot{q} = [-0.1087 \quad 0.4428]^T$ rad/s, $\dot{q}_e = [-0.2087 \quad 0.2428]^T$ rad/s,

$W_{ne} = -0.0311$ N. m, $\Delta T = -0.1112$ N. m

The results show that the classical method produces an energy increase and the use of Newton's coefficient of restitution with Routh's method does not resolve this inconsistency. To elaborate more about the energy dissipation from the three coefficients of restitution, we consider $e = 1$, which means that the dissipation of energy comes only from the friction forces. The three coefficients of restitution give:

The Newton's coefficient of restitution

$\dot{q}_e = [-0.3353 \quad 0.4977]^T$ rad/s, $\Delta T = 0.1305$ N. m, $W_{te} = -0.2122$ N. m

The Poisson's coefficient of restitution

$\dot{q}_e = [-0.2114 \quad 0.2482]^T$ rad/s, $\Delta T = -0.1075$ N. m, $W_{te} = -0.0808$ N. m

The energetic coefficient of restitution

$\dot{q}_e = [-0.2261 \quad 0.2779]^T$ rad/s, $\Delta T = -0.0860$ N. m, $W_{te} = -0.0860$ N. m

Therefore, Newton's coefficient results in an energy gain upon impact, violating the laws of physics. Although Poisson's coefficient does not lead to an energy increase, it introduces energy dissipation through non-frictional forces, which also contradicts physical laws. In contrast, the energetic coefficient adheres to the laws of physics, with energy dissipation arising solely from the frictional component of the impact.

4. Conclusions

A mathematical model and simulation technique for multibody systems undergoing impact with friction has been developed. The frictional behavior is modeled using Coulomb's law with stiction, while the coefficient of restitution is employed to represent the plasticity of collisions. To ensure the accurate application of Coulomb's law, Routh's incremental method is utilized. A critical coefficient of friction that dependents solely on the system's inertia and orientation, has been identified. Equations of motion for both sticking and sliding phases are formulated using the normal impulse at the collision point as the independent variable, and three distinct definitions of the coefficient of restitution are introduced to characterize the termination of impact. During three-dimensional impacts, the contact point may change its sliding direction, transition to a sticking phase, or experience an instantaneous stop followed by sliding in a new direction without swerving. The equations governing these behaviors are expressed as a system of nonlinear ordinary differential equations, for which exact analytical solutions are generally unattainable; hence, numerical methods are employed. A flowchart outlining the numerical procedure is provided, accounting for all potential contact scenarios. For planar impacts, analytical solutions to the equations of motion have been derived. Five distinct contact modes are identified, and the specific conditions—based on system inertia, orientation, initial velocity, and the coefficient of restitution—that lead to each mode are defined. Closed-form solutions are presented for each mode, enabling the computation of the final velocity at the contact point, the resulting impulse, and the change in generalized joint velocity. Kan's example is analyzed to demonstrate the effectiveness of the proposed formulation. When solved using traditional textbook methods, the case exhibits an increase in kinetic energy, violating the laws of physics. Using the algebraic solutions developed in this paper, results are obtained and compared across three definitions of the coefficient of restitution. A special case where no energy dissipation occurs in the normal direction (i.e. $e = 1$) is also examined. Among the tested coefficients, only the energetic coefficient conforms to the laws of physics. In contrast, both Newton's and Poisson's coefficients result in violations—either through an increase in energy or dissipation in the normal direction, respectively.

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